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Laplacian Eigenmaps and Geometric Structures of Compact Manifolds

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Abstract of Report Talk: Laplacian eigenmaps provide useful insights into dimension reduction problems in data analysis and machine learning. Much of the work utilizing discrete graph Laplacians has been motivated by Belkin and Niyogi (2005), who demonstrated how to use Laplacian eigenmaps to preserve the local geometry of a manifold $\mathcal{M}$.
In this talk we investigate, theoretically and numerically, eigenmaps of continuous Laplacians on compact manifolds and explore how they can affect the global geometric structure of $\mathcal{M}$. We start with the simplest case when $\mathcal{M}$ is the unit interval $[0,1]$. Initially we focus on the eigenmap derived from the first two lowest non-zero eigenvalues to ensure smooth variations – useful for preserving the local geometry of $\mathcal{M}$. We justify that this eigenmap in $\mathbb{R}^2$ yields a parabolic shape due to the quadratic relationship in the eigenfunction corresponding to the second non-zero eigenvalue.
We then analyze the rectangle $[0,1]\times[0,a]$ and discuss how varying a affects the eigenfunctions and their maps. We prove that if $0 < a < \frac{1}{2}$, the eigenmap behaves similarly to that of the interval; if $\frac{1}{2} < a \leq 1$, the eigenmap closely approximates a rectangular shape; and at the critical point $a = \frac{1}{2}$, the map alternates between parabolic and rectangular shapes. The main topic of our research is a similar study for the flat torus $[0,1] \times [0,a]/\sim$ which substantially improves the previous results.

Singular Laplacian eigenmaps via Green's function

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Abstract of Report Talk: Laplacian eigenmaps are prevalent in dimension reduction algorithms, yet the theory behind these eigenmaps remains insufficient to explain their accuracy and applicability. Although effort has gone into improving estimates of the continuous Laplace-Beltrami operator and the discrete graph Laplacian, attention to their inverse operators has been marginal.
In our work, we take a step into filling this gap. We begin by constructing graphs from n points placed deterministically and randomly over the unit interval. Following a paper written by Croydon (2017), we consider graph conductance and resistance and derive a resistance metric. With this, we use an associated formula from the paper for the Green's function and thus the inverse Laplacian. Following a similar process for the Euclidean metric, we find the expected Green's function. Finally, we compare these results and analyze the accuracy of this method. We use the Law of Large Numbers to predict the behavior of random models as n becomes large. We conclude the talk with analysis of the variance of the random graphs and show that this method will have better accuracy than the existing theory.

New Geo-Arithmetical and Polynomial Results in Ramsey Theory

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Abstract of Report Talk: A classical result in Ramsey theory (and one of Khintchine's *Three Pearls of Number Theory*) is van der Waerden's theorem (1927), which states that for any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, one of the C_i contains arbitrarily long arithmetic progressions.

In 2005, Professor Vitaly Bergelson added a multiplicative twist to van der Waerden's theorem by proving that for any finite partition of $\mathbb{N}$ and any $k \in \mathbb{N}$, there exist $a, b, d \in \mathbb{N}$ for which $\{b(a + id)^j : 1 \leq i, j \leq k\}$ is contained in a single cell of the partition. We enhance his results and show (among other things) that for any finite partition of $\mathbb{N}$ and any $k \in \mathbb{N}$, there exist $b, x_1, ..., x_k, d \in \mathbb{N}$ for which the configuration

$$\{b(x_1 + i_1 d)^{j_1}(x_2 + i_2 d)^{j_2} \cdots (x_k + i_k d)^{j_k} : 1 \leq i_1, ..., i_k, j_1, ..., j_k \leq k\}$$
$$\cup\{(x_1 + i_1 d)(x_2 + i_2 d) \cdots (x_k + i_k d) : 1 \leq i_1, ..., i_k \leq k\} \cup \{d\}$$

is contained in a single cell of the partition. Such configurations are called "geo-arithmetic" since they contain both arithmetic and geometric configurations. As a bonus, the ergodic-theoretic methodology that we employ is general enough to yield analogous results over other rings such as $\mathbb{F}_q[t]$ and $\mathbb{R}$.
We also "polynomialize" some of Dr. Bergelson's results from 2005 and obtain a sharpened form of the polynomial Hales-Jewett theorem along the way.

Convergence Analysis for Edge-Informed Gaussian PSF Approximations

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Abstract of Report Talk: Signal measurement processes often suffer from environmental and instrumental limitations, leading to unavoidable blurring in captured signals – typically modeled as a convolution between the underlying signal or function and a *point spread function* (PSF) that characterizes the blurring artifact. Standard deblurring algorithms assume a known PSF, but in many cases, both the PSF and the signal are unknown, with only limited, blurred data available, necessitating accurate PSF recovery. We utilize a method that applies convolutional edge-detection kernels (introduced by Gelb and Tadmor (1999)) to approximate a Gaussian PSF given a blurred piecewise-smooth signal. Through an approximation theoretic approach, we prove that certain classes of the Gelb-Tadmor kernels applied to representative blurred piecewise-constant and linear signals result in at least quadratic convergence of PSF approximations with respect to the L^2 norm. We also show convergence to a small constant bound for PSF approximations obtained from a generalized family of higher-order, piecewise signals. Moreover, we extend our analysis to blurred signals with additive white noise and simple images.

Markowitz Optimal Portfolio Theory: Risk Measures

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Abstract of Report Talk: In 1952, Harry Markowitz introduced the Nobel Prize-winning *Markowitz Optimization Model* — a constrained optimization problem which aids investors in finding the *optimal* allocation of capital ω_k among assets for $k = 1, 2, \ldots, n$ in a portfolio. The model maximizes the % *the most optimal weight distribution,* ω, for a portfolio through maximizing expected portfolio return $\%(\overline{\mu})$ subject to the risk — *variance* $\%(\sigma^2)$ — of the portfolio being at a certain level σ^2:

$$\begin{aligned} \max_{\omega_K} \quad & \sum_{k=1}^{n} \overline{\mu}_k \omega_k, \\ \text{s.t.} \quad & \sum_{k=1}^{n} \sum_{j=1}^{n} \omega_k \omega_j \sigma_{\mu_k,\mu_j} = \sigma^2, \sum_{k=1}^{n} \omega_k = 1, \end{aligned}$$

where $\overline{\mu}_k$ is the expected return of asset k, and σ_{μ_k,μ_j} is the covariance of the returns of assets k and j.

After further analysis of the Markowitz Optimization Model, we explored extensions of the model through the consideration of novel risk measures — particularly, *Tail Value at Risk (TVaR)*, which adheres to more modern nuances of risk in comparison to variance.

To assist us in our computations, we implemented a Python program which solves the constrained optimization problem through a given set of inputs. These computations will utilize real-world data in order to further compare the various risk measures and evaluate their performances— allowing us to investigate how capital is allocated among assets under the Markowitz framework with various risk measure preferences.

Counterexamples to the Boninger results about Jones type polynomials for links in thickened surfaces

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Abstract of Report Talk: In 1984, Morwen Thistlethwaite introduced a polynomial $\Gamma_G(t)$ defined for signed graphs G, which parallels the Jones polynomial invariant for classical knots. Although defined for general signed graphs G, Thistlethwaite noted that no application for $\Gamma_G(t)$ is known in the case of non-planar G. In 2022, Joe Boninger used $\Gamma_G(t)$ in its full generality to construct a pair of polynomials defined for checkerboard colorable knots on thickened surfaces. Boninger goes on to claim that his pair of polynomials is a knot invariant. However, through a deep analysis of Boninger's proof, we have found a logical error. We construct explicit counterexamples that show Boninger's statement to be incorrect. These counterexamples naturally lead to a method of mending the result. To this end, We introduce the concept of generalized ribbon graphs and we establish their relationship with knots on thickened surfaces. This framework suggests a more general formulation of Boninger's result, grounded in the theory of generalized ribbon graphs.

A Generalization of Merton's Investor Problem

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Abstract of Report Talk: In 1969, economist Robert C. Merton published a continuous-time model of portfolio selection that examined how an investor should divide their wealth between stocks, bonds, and consumption in order to maximize their expected lifetime utility. This model assumes geometric Brownian motion to model stock prices and uses stochastic calculus to derive the budget equation which constrains the investor.

We aim to generalize this problem by incorporating occupational income with a punishment to utility as income increases, changing the model to the following:

$$\max_{c(t),I(t),\omega(t)} \mathbb{E}\Big[\int_0^T U[X(s)c(s),X(s)I(s)]ds + B(X(T))\Big]$$

such that

$$dX_t = \sigma\omega(t)X_t dW_t + ((\omega(t)(\alpha - r) + r) - c(t) + I(t))X_t dt,$$

where the utility function $U[X(s)c(s),X(s)I(s)]$ takes into account both an increase in utility from consumption and a decrease in utility for earning income. We use dynamic programming to reduce the constrained maximization problem into solving a partial differential equation equipped with suitable initial conditions (an initial value problem). We then solve the initial value problem using modern numerical analysis techniques.

Topological Correspondance Between the Hausdorff Distance and a Quasi-Metric

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Abstract of Report Talk: In convex geometry, the classical Hausdorff distance d_H turns the space of compact, convex subsets of $\mathbb{R}^n$ contained in the unit simplex into a compact metric space. Alternatively, Darvas-Di Nezza-Lu define a quasi-metric $d_{\tilde{\mathcal{C}}}$ on this space using mixed volumes, inspired by a construction in complex geometry. We show that $d_{\tilde{\mathcal{C}}}$ generates a metrizable topology equivalent to d_H. Using convex geometric means, we also establish the Hölder estimate

$$c \cdot d_H(K, L)^n \leq d_{\tilde{\mathcal{C}}}(K, L) \leq C \cdot d_H(K, L)$$

for convex sets K and L contained in the unit simplex and $C, c > 0$ are constants depending only on n. We investigate the role of the unit simplex in the definition of $d_{\tilde{\mathcal{C}}}$, and show that the optimality of the Hölder exponent is closely related to the boundary structure of the simplex.

Braiding on Categories of Superselection Sectors

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Abstract of Report Talk: A superselection sector (SSS) is a way of representing a net of algebras over some partially ordered set $\mathcal{P}$. Gabbiani and Fröhlich have previously proven that when $\mathcal{P}$ is the poset of intervals on S^1, there is a strict braided (commutativity constraint) tensor category in which the objects are SSS localized at a fixed $p \in \mathcal{P}$ and the morphisms are intertwiners. Our work aims to develop analogous results for other posets, such as the poset of cones in $\mathbb{R}^2$, by providing axioms that posets must satisfy to get a strict braided tensor category of SSS.

Special values of Bessel Functions for subgroups of GL_n

David K.R. Cates (dkrc@tamu.edu)
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Abstract of Report Talk: The study of the Local Langlands program requires understanding the representations of $\mathrm{GL}_n(F)$ and its subgroups, where F is a local field. One way to approach this is to find methods that work for F a finite field, rather than a local field. A common idea is to associate a "Gamma factor" with a pair of representations, which we can compute, giving you additional information about how the representations are related to one another. We use the Langlands-Shahidi method to define the Gamma factor for a pair of irreducible representations of the groups $\mathrm{Sp}_{2n}(F), \mathrm{O}_n(F)$, and $U_n(F)$, and show that it is multiplicative. As a corollary, we compute special values of the Bessel function associated with these representations.

Extending the support for n-th centered moments of a family of automorphic L-functions

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Abstract of Report Talk: The Katz-Sarnak philosophy predicts that the distribution of zeros near the central point of a family of L-functions agrees with the distribution of eigenvalues near 1 of a classical compact group depending on the symmetry type of the family. Katz and Sarnak introduced the 1-level density to study the arithmetically important zeros near the central point using smooth counting functions which serve as test functions. Assuming the generalized Riemann Hypothesis (GRH), Iwaniec, Luo, and Sarnak, established agreement between the 1-level density of a family of automorphic L-functions and the density predicted by Katz and Sarnak for test functions whose Fourier transform is supported in $(-2, 2)$.
Under GRH, Baluyot, Chandee, and Li proved that the density predicted by Katz and Sarnak holds for an extension of the family studied by Iwaniec, Luo, and Sarnak in the range $(-4, 4)$, nearly doubling previously best known results. We generalize their main techniques to the study the higher centered moments of the 1-level density of this larger family, leading to better results on the behavior near the central point. Numerous technical obstructions emerge that are not present in the 1-level density. Averaging over the level and assuming GRH, we prove the density predicted by Katz and Sarnak holds for the n-th centered moments for test functions supported in $\sigma = \min\left\{3/2(n-1), 4/(2n - \mathbf{1}_{2\nmid n})\right\}$. We prove the 2-level density agrees with the Katz-Sarnak density conjecture for test functions supported in $\sigma_1 = 3/2$ and $\sigma_2 = 5/6$, respectively, extending the previous best known sum of supports for the 2-level density. This work is the first evidence of an interesting new phenomenon: by taking different test functions, we are able to extend the range in which the Katz-Sarnak density predictions hold. For $n = 3$, our results improve the previously best known uniform support from $\sigma = 2/3$ to $\sigma = 3/4$. The techniques we develop can be applied to understanding quantities related to this family containing sums over multiple primes.

Patterns of Primes in Joint Sato–Tate Distributions

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Abstract of Report Talk: For $j = 1, 2$, let $f_j(z) = \sum_{n=1}^{\infty} a_j(n)e^{2\pi i n z}$ be a holomorphic, non-CM cuspidal newform of even weight $k_j \geq 2$ with trivial nebentypus. For each prime p, let $\theta_j(p) \in [0, \pi]$ be the angle such that $a_j(p) = 2p^{(k-1)/2} \cos\theta_j(p)$. The now-proven Sato–Tate conjecture states that the angles $(\theta_j(p))$ equidistribute with respect to the measure $d\mu_{ST} = \frac{2}{\pi}\sin^2\theta\, d\theta$. We show that, if f_1 is not a character twist of f_2, then for subintervals $I_1, I_2 \subset [0, \pi]$, there exist infinitely many bounded gaps between the primes p such that $\theta_1(p) \in I_1$ and $\theta_2(p) \in I_2$. We also prove a common generalization of the bounded gaps with the Green–Tao theorem.

Stable Subsets of Groups

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Abstract of Report Talk: Historically, model theory and combinatorics have been deeply intertwined, with qualitative theorems in model theory often having quantitative combinatorial analogs. In particular, the notion of a formula being stable is equivalent to a certain graph containing no copies of the full half-graph. Motivated by this equivalence, we may call a a graph k-stable if it does not contain a half-graph of height k. Stable graphs are often thought to have "nicer" combinatorial properties and so stability is often studied in the context of other combinatorial problems. For instance, Malliaris and Shelah showed that any stable graph permits a regular partition of size polynomial in its degree of uniformity, while the lower bounds (due to Gowers) on the size of the partition in Szemeredi's regularity lemma are of tower type.
In this talk, we first introduce the the notion of a stable subset of a group and then investigate two properties of stable sets particular to the group setting: one involving genericity of stable subsets of groups and other involving arithmetic progressions in stable subsets of the natural numbers.

Uniform distribution and computability theory: UD randomness

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Abstract of Report Talk: We are working at the intersection of computability theory and analysis to study UD-random real numbers. We use computable functions: those that can be defined using a computer program that halts on every input.
A sequence is uniformly distributed (UD) on the interval $[0, 1]$ if the frequency at which points appear in every subinterval of $[0, 1]$ is proportional to the length of the subinterval. In 2013, Avigad defined a UD-random real x as one for which the sequence $\langle [a_i \cdot x] \rangle$ is UD for every computable sequence $\langle a_i \rangle$ of distinct nonnegative integers.
We study the randomness of n-computably enumerable (c.e.) subsets of $\mathbb{N}$: sets defined by allowing a computer to "change its mind", up to n times, on which natural numbers belong in our set. Formally, if we take a computable function $f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$, whose limit over s is the characteristic function of our set, we will find that the output $f(k, s)$ will change at most n times for any given k. We then use the characteristic function χ_A to represent A as the real number $.\chi_A(0)\chi_A(1)\chi_A(2)\ldots$. We prove that n-c.e. sets are not UD random for any n.

Random Matrix Theory for Symmetric Matrices

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Abstract of Report Talk: A classical result of random matrix theory (approximately) states that the kth powers of the traces of $n \times n$ unitary matrices converge in distribution to independent Gaussian distributions. We are interested in analogous results over the field $\mathbb{Z}/p\mathbb{Z}$ for odd primes p. For example, Ofir Gorodetsky and Brad Rodgers have shown equidistribution results for powers of traces of $n \times n$ invertible matrices with entries in $\mathbb{Z}/p\mathbb{Z}$, and their work also handles other matrix groups. In our work, we have shown that the trace and determinant of $n \times n$ invertible symmetric matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ jointly equidistribute, and our results extend to the set of invertible alternating matrices. Our techniques involve certain "matrix Gauss sums" over symmetric and alternating matrices, which we evaluate via an inductive approach on the dimension n.

Homomesy and Parking Functions

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Abstract of Report Talk: Given a finite set A, homomesy studies the average values of a discrete statistic on A over the orbits of a bijection from A to A. In [Elder, et al. 2023], new instances of homomesy for permutations were discovered by searching the Findstat database for bijection-statistic pairs. We extend their work, by considering the set of parking functions and some specific subsets of this set. We also introduce a new subset of parking functions, called early preference parking functions, for which we prove an additional case of homomesy that is not observed in parking functions.

Optimal Design Problems with Cost Function: Regularity of Minimizers and Free Boundary Analysis

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Abstract of Presentation: We investigate optimal design problems with non-constant cost functions, where certain regions are more costly to occupy than others. This leads to the analysis of a complex free boundary problem of the Alt-Caffarelli type. We show the existence of solutions u to the cost problem and establish the regularity of minimizers and their free boundaries. In particular, we show u is universally Hölder continuous if the cost function $c(x) \in L^p$, with $p > \frac{n}{2}$. For $p = \infty$, we achieve Lipschitz continuity of minimizers. A free boundary condition is proven to hold via a geometric measure Hadamard's variational formula. Finally, we establish the uniqueness and stability properties of the solution when the body to be insulated is star-like.

Toric Surface Codes and the Periodicity of Polytopes

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Abstract of Report Talk: Toric codes, originally introduced by Hansen as a generalization of Reed-Solomon codes, are k-dimensional subspaces of $\mathbb{F}_q^n$ derived from toric varieties that correspond uniquely to integral convex polytopes in $\mathbb{R}^m$ with k lattice points. In previous work, Soprunov and Soprunova (2009) used the *Minkowski length* $L(P)$, an important geometric invariant defined as the largest number of primitive line segments whose Minkowski sum is contained in a polytope P, to bound the minimum distance of P's corresponding toric code. In our work, we study what we define as *period-1 polytopes* – polytopes satisfying the property $L(tP) = tL(P)$ for all $t \in \mathbb{Z}^+$, where tP is the t-dilate of P. We begin by identifying the periodicity of certain classes of polytopes, typically by proving explicit formulas for their Minkowski length. We then prove a correspondence between codewords with minimal Hamming weight and maximal decompositions within P. Using this new correspondence, we prove an explicit formula for the minimum distance of toric codes associated to a class of period-1 polytopes.

Second Moment Calculations for One-Parameter Families of Elliptic Curves

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Abstract of Report Talk: For a fixed elliptic curve E without complex multiplication, the distribution of $a_E(p)|E[\mathbb{F}_p]| - (p+1)$ converges to a semicircular distribution. Michel proved that for a one-parameter family of elliptic curves $y^2 = x^3 + A(T)x + B(T)$ with $A(T), B(T) \in \mathbb{Z}[T]$ and non-constant j-invariant, the second moment of the Fourier coefficients of the elliptic curves $a_{E_t}(p)$ is $p^2 + O(p^{3/2})$; the rate of convergence has applications to the distribution of zeros near the central point and the Birch and Swinnerton-Dyer conjecture.
S. J. Miller conjectured that the highest order term of error of the second moment that does not average to zero is on average negative, which has implications for the observed excess rank in such families. For one-parameter families where the polynomials are linear or quadratic and certain cubics and quartics, the second moment can be computed explicitly using sums of Legendre symbols. However, these techniques do not generalize to higher degree polynomials and new approaches are needed; further, most of the families investigated in previous work are special families and thus their behavior may not be generic. To this end, we computationally investigate biases in the second moment of one-parameter families where Legendre symbol calculations are not tractable. We use efficient algorithms and storage (Cipolla's algorithm and taking advantage of some automorphisms of elliptic curves) to quickly compute $a_E(p)$ values and develop statistics to isolate lower order terms in the second moment expansion. We also explore the connections between the moments of a one-parameter family and counting points modulo p on rational threefolds using techniques from algebraic geometry.

Minimum Polyform Puzzle

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Abstract of Report Talk: A polyomino is a connected shape made by gluing together identical squares called cells. For a given polyomino p and positive integer n, we are interested in the smallest polyomino P containing n copies of p, allowing for overlaps. We let $a_{p,n}$ denote the number of cells in P.
We establish that for fixed p, the sequence $\{a_{p,n}\}$ is asymptotically linear. We show bijectively that there are classes of polyominoes whose minimum sizes $a_{p,n}$ are equivalent. Furthermore, there are transformations which demonstrate for certain pairs of polyominoes p and q that $a_{p,n} \geq a_{q,n}$ for all n.
For the square tetromino, we show with integer programming that $a_{p,n} = \lceil n + \sqrt{4n} + 1 \rceil$. We conjecture similar closed form solutions for polyforms with up to 12 cells, as well as an algorithm for constructing such shapes.
These results have implications for "de Bruijn polyominoes," which are the smallest colored polyominoes containing each k-coloring of a given polyomino p. We introduce novel constructions of de Bruijn polyominoes and other polyforms that follow from the aforementioned work. This generalizes earlier research into de Bruijn sequences, arrays, and tori.

Computing Galois Actions of Torsion on Abelian Varieties

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Brown University [Mentor:Isabel Vogt]

Abstract of Report Talk: An abelian surface is a compact algebraic surface endowed with a compatible abelian group structure. They are two-dimensional analogues of elliptic curves. Of particular interest are the torsion points on such surfaces. Over the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, the ℓ-torsion points form a subgroup isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^4$. For prime ℓ, this is a vector space over $\mathbb{F}_\ell$. The Galois automorphisms $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permute the ℓ torsion, giving a Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\mathrm{GL}(4, \ell)$. The image is contained in the general symplectic group $\mathrm{GSp}(4, \ell)$ and typically surjects onto GSp. When the image is smaller, this indicates extra, non-generic arithmetic structure on the surface. It is already known, given a surface, how to probabilistically compute whether the image of Galois is surjective for a given ℓ. We probabilistically compute the precise $\ell = 5$ image of Galois for over 800 of the 939 surfaces with typical endomorphism ring in the L-functions and Modular Forms Database (LMFDB) for which the $\ell = 5$ image of Galois is not surjective. For the remaining surfaces, we determine the order of the image and give a list of (at most four) candidate images.

Minimum Distances of an Infinite Class of Toric Surface Codes

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Abstract of Report Talk: In the field of coding theory, finding the minimum distance of a linear code is vital, as it determines a code's error correction capabilities. Finding this value is especially challenging for toric codes, a generalization of Reed-Solomon codes introduced by Hansen in 1997. Toric codes are k-dimensional subspaces of $\mathbb{F}_q^n$ obtained from toric varieties, which have a unique correspondence to integral convex polytopes in $\mathbb{R}^n$. Little and Schwarz (2007) outlined an elementary method using Vandermonde matrices to determine minimum distance formulas for toric codes given by simplices and boxes. To utilize this method for other polytopes, we first define a *staircase configuration*, a special collection of points in $(\mathbb{F}_q^*)^2$. We then prove that every Vandermonde matrix associated to a polytope from a certain class in $\mathbb{R}^2$ has non-zero determinant when evaluated at an appropriate staircase configuration. This class is described for any $\ell \in \mathbb{Z}^+$ by $\ell[0,(0,1)] + \ell[0,(1,0)] + \ell\Delta$ where $+$ denotes the Minkowski sum and Δ denotes the standard 2-simplex. Finally, we prove explicit minimum distance formulas for this infinite class of polytopes.

Congruence Classes of Simplex Structures in Finite Field Vector Spaces

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Abstract of Report Talk: The Erdős-Falconer distance problem over finite fields asks how large a subset $E \subset \mathbb{F}_q^d$ needs to be so that pairs of points in E determine all possible distances in $\mathbb{F}_q$, or more generally determine a positive proportion of these distances. More precisely, we define the distance between two points $x, y \in \mathbb{F}_q^d$ as

$$||x - y|| = (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2$$

and we want to find the infimum of $s \in \mathbb{R}$ such that E determines a positive proportion of distances in $\mathbb{F}_q$ whenever $|E| \gtrsim q^s$. While this distance function is not a norm, it is still a natural quantity to study as it is invariant under orthogonal transformations of $\mathbb{F}_q^d$.
A natural generalization of this question is to consider a graph G and say that two embeddings $p, p' : V(G) \to \mathbb{F}_q^d$ of a graph G are congruent if for every $(v_i, v_j) \in E(G)$ we have that $||p(v_i) - p(v_j)|| = ||p'(v_i) - p'(v_j)||$. What is the infimum of s such that E contains a positive proportion of congruence classes of G in $\mathbb{F}_q^d$ whenever $|E| \gtrsim q^s$? For G a complete graph, whose embeddings correspond to simplices in $\mathbb{F}_q^d$, Bennett, Hart, Iosevich, Pakianathan, and Rudnev were able to prove bounds on s using group-action methods that take advantage of the rigidity of G. For graphs with less rigidity such as paths, trees, and cycles, success has been found in using techniques that take advantage of the inductive nature of these graphs.
Recently, Aksoy, Iosevich, and McDonald were able to combine these two approaches to prove nontrivial bounds on the "bowtie" graph - two triangles joined at a vertex. We extend their results by developing a framework to handle a wide class of graphs that exhibit a combination of rigid and loose behavior. Our proof relies on an application of the Hadamard Three Lines Theorem to reduce our analysis to a simple class of simplex structures. In $\mathbb{F}_q^2$, this approach give new nontrivial bounds on trees of simplices. In $\mathbb{F}_q^d$, our methods surpass the best known bounds for chains of high-dimensional simplices and structures with high vertex degree. We also discuss current progress on how this framework can be extended to more general simplex structures, such as cycles of simplices and structures of simplices glued together along an edge or a face.

Structural Characterizations of Well-Edge-Dominated Graphs

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Abstract of Report Talk: A graph G is well-edge-dominated if all of its minimal edge-dominating sets have the same cardinality. In 1980, Yannakakis and Gavril showed that the edge-dominating set problem is NP-complete even when restricted to bipartite or planar graphs. Therefore, a structural characterization is necessary for identifying well-edge-dominated graphs efficiently. Recently, Anderson et al. (2021) showed that for graphs of girth at least 4, meaning the smallest cycle in G has order greater than or equal to 4, any well-edge-dominated graph is either bipartite or a member of $\{C_5, C_7, H^*\}$ where H^* is C_7 with a chord. Therefore, the last piece of the puzzle is characterizing all well-edge-dominated graphs containing triangles and well-edge-dominated bipartite graphs containing 4-cycles. Our research extends this to a partial characterization of graphs containing exactly one triangle. In particular, we show that if a well-edge-dominated graph G contains exactly one triangle, then by carefully removing specific closed edge neighborhoods, the remaining components must be well-edge-dominated bipartite graphs which satisfy a list of properties. Additionally, we provide infinite classes of well-edge-dominated graphs which contain multiple triangles, showing that the class containing triangles is nontrivial. It is our hope that we will fully characterize the well-edge-dominated graphs containing exactly one triangle as a first step in characterizing all such graphs with girth 3. Finally, we show that the line graph of a well-edge-dominated graph could be used to construct a potential counterexample to Vizing's conjecture, a very important and long standing conjecture in domination theory.

Bouncing Outer Billiards

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Abstract of Presentation: We describe and analyze a new class of dynamical systems which we call bouncing outer billiards. They are generalizations of the outer billiards introduced by Neumann in 1950s . First, we give precise description of dynamics, including all periodic orbits, of this system on the line segment. We prove the existence of fixed points for general bouncing outer billiards systems. Furthermore, we study measure-based properties of the system, and demonstrate examples of complex behavior exhibited with the aid of computer simulation.

Recursion for the Partial-Dual Euler Genus Polynomial

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Abstract of Report Talk: In topological graph theory, ribbon graphs are surfaces with boundary made up of vertex-discs and edge-ribbons and are equivalent to graphs cellularly embedded in a surface. Geometric duality is an operation that swaps the roles of the vertices and faces of a ribbon graph. Chmutov defined partial duality as the application of geometric duality to a spanning ribbon subgraph. Gross, Mansour, and Tucker introduced the partial-dual Euler genus polynomial, a generating function that counts partial duals of a ribbon graph by their Euler genus.

Recently, Yan and Jin showed that if two bouquets (one-vertex ribbon graphs) have the same signed intersection graph, then they have the same partial-dual polynomial. Thus the intersection polynomial of a signed intersection graph G can be defined as the partial-dual polynomial of any bouquet whose signed intersection graph is isomorphic to G.

We derive a recurrence relation for the intersection polynomial at cut-vertices, generalizing a recursion by Yan and Jin that applies to leaves. Incidentally, the twist polynomial defined by Yan and Jin for delta-matroids defines a polynomial on the class of simple signed graphs that is identical to the intersection polynomial when restricted to signed intersection graphs. Our recurrence relation extends to this more general setting. An interesting direction for future work is to find a completely recursive definition for the intersection polynomial or twist polynomial on simple signed graphs (analogous to the deletion-contraction relations of the Tutte polynomial).

Benford's Law and Multinomial Distributions in Stick Fragmentation

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Abstract of Report Talk: In the late 19th century Newcomb observed that particular pages of logarithmic tables were more worn than others, highlighting how the leading digits were not equally distributed; 50 years later Benford rediscovered this phenomenon and published the frequency of leading digits of over 20,000 observations, spanning a multitude of data sets. We say a dataset follows Benford's Law base B if the probability of a leading digit d is $\log_B(\frac{d+1}{d})$; for base 10 we have 1 is the leading digit almost 30.1% of the time, 2 around 17.6%, dwindling to 9 at about 4.6%.
Benford's Law emerges in a variety of data sets, from geophysics and nature to astrophysics and mathematical constants and functions, and has tremendous success in detecting fraud and data integrity issues. Thus, it is extremely valuable to understand what types of processes lead to Benford or non-Benford behavior. While most theoretical results involve independent variables, Becker et al. examined Benford behavior involving dependent random variables in fragmentation problems by introducing a tree network and splitting the analysis based on the first shared ancestor. Most of the paper follows cutting a stick of length L at proportion p, where $p \in (0, 1)$. The first cut results in two sticks, and is repeated for N iterations leading to 2^N sticks with $N + 1$ distinct lengths. We extend their results to multiple fixed cuts, for example cutting each piece at proportion p and q each time, resulting in three segments of lengths p, q, and $(1-p-q)$ after 1 iteration and $(N+1)(N+2)/2$ segments for N iterations. We prove that there are special choices of p and q that lead to non-Benford behavior; the proof relies on algebraic properties of logarithms of certain irrationals and combinatorics of multinomial coefficients. If time permits we discuss when the behavior is Benford.

Information Diffusion on Iterated Graph Models

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Abstract of Report Talk: We explore a particular discrete time process, zero forcing, on graphs which model social networks. The Iterated Local Transitivity (ILT) model, introduced in 2009 by Bonato et al., models online social networks and captures the notion that friends of friends are likely friends. Later in 2017, Bonato et al. expanded this idea to the Iterated Local Anti-Transitivity ($ILAT$) model which captures the notion that enemies of enemies are likely friends. Both the ILT and $ILAT$ models exhibit properties observed in real-world complex networks, such as distances bounded by absolute constants and bad spectral expansion.
The zero forcing process, first introduced in 2008 by Barioli et al., is applicable to the minimum rank problem from linear algebra, graph searching algorithms, and determining how fast information can spread in a network—a natural consideration for graphs modeling social networks. It is defined as follows: Let $S \subseteq V(G)$ where each $v \in S$ is called forced, and each $u \in V(G) \setminus S$ is called unforced. Let $U := V(G) \setminus S$. If $v \in S$ has exactly one neighboring vertex $u \in U$, then v "forces" u, i.e. u is removed from U and added to S. If eventually every vertex in our graph is forced, we say our initial set S is a zero forcing set; otherwise, we say S is a failed zero forcing set. The failed zero forcing number, formally introduced by Fetcie et al. in 2015, is the maximum cardinality of any failed zero forcing set. We focus on bounds for the failed zero forcing number of all graphs constructed using the ILT and $ILAT$ frameworks. We obtained these bounds by examining the neighborhoods of finitely many vertices in arbitrary ILT and $ILAT$ graphs. We then made appropriate choices, based on the relationships between specific vertices inherent to these models, of which vertices to include in S such that S would constitute a failed zero forcing set of maximal size.

Long Paths in Polynomial Divisor Graphs

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Abstract of Report Talk: The divisor graph $D(n)$ has vertices $\{1, 2, ..., n\}$, and edge between vertices v, u if $v|u$ or $u|v$. Prior work of Pollington, Pomerance, Tenenbaum, and Saias has investigated the length $f(n)$ of the longest path in this divisor graph. After the work of Saias, it is known there exists constants c_1, c_2 such that, for sufficiently large n, $f(n)$, can be bounded by

$$c_1 \frac{n}{\log n} \leq f(n) \leq c_2 \frac{n}{\log n}. \tag{1}$$

We investigate the analogous question for polynomials over a finite field $\mathbb{F}_q$ of order q. The divisor graph $D_q(n)$ has vertices for each monic polynomial in $\mathbb{F}_q[x]$ of degree at most n, and an edge between vertices F, G if $F|G$ or $G|F$. Generalizing the work of the above authors, and using techniques from the theory of function fields we establish bounds analogous to those in (1) for $f_q(n)$, the length of the longest path in $D_q(n)$. In particular, we show there exists constants k_1, k_2, such that

$$k_1 \frac{q^n}{n} \leq f_q(n) \leq k_2 \frac{q^n}{n}.$$

Future work would establish explicit values for the constants k_1, k_2.

Symbolic Dynamics on Countable Alphabets

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Abstract of Report Talk: We define and discuss specification properties for countable state shift spaces, which are special cases of definitions from an upcoming paper by Climenhaga, Thompson, and Wang and generalize the well-studied compact specification property to non-compact shift spaces. We present an infinite class of examples of such shift spaces and prove the variational principle for these spaces. This gives the foundations for developing the theory of equilibrium states in this new setting.

Elliptic Curves and Bounding the 7-Torsion of Ideal Class Groups of Imaginary Quadratic Fields

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Abstract of Summary Talk: Since Gauss presented his class number problem for negative discriminants D, people have been interested in the behavior of class numbers $h(D)$ as $D \to -\infty$, which is the size of the class group $C(D)$, a finite abelian group of great interest in number theory. Because this problem has proved to be so difficult, there is also a long tradition of attention to variants of the class number problem, such as bounding the size $h_p(D)$ of the p-torsion subgroup of $C(D)$ as $D \to -\infty$. Shankar and Tsimerman developed a new approach to produce bounds on $h_p(D)$ using twists of elliptic curves and the Birch and Swinnerton-Dyer conjecture. However, their methods only work for $p = 2, 3$, and 5 since "splittability," a property of the short exact sequence involving the p-torsion of the elliptic curve as a Galois module, holds only for these primes. We present our work on bounding the defect that arises when $p = 7$ and thus splittability fails, looking at the statistical behavior of the connecting map in the long exact sequence in Galois cohomology. This is joint work with Carl Wang-Erickson.

Generalized Movement Operators for Non-Abelian Anyon Theories

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Abstract of Report Talk: Given a finite group, G, one can construct a lattice model with a commuting projector Hamiltonian, as per Kitaev's *quantum double* construction. The low energy effective field theory of this model is a topological quantum field theory whose excitations are referred to as *anyons*. The different types of anyons of a quantum double model are given exactly by the irreducible representations of the Drinfeld double of the group, $D(G)$.
One can use local operators acting on the lattice to create, move, and split these anyons. While movement operators for the abelian quantum double model are well understood, the same cannot be said for the general non-abelian case. We present the first unitary movement operators for the quantum double model for an **arbitrary** finite group. Surprisingly, these movement operators can be used to move any given anyon, regardless of which representation it corresponds to.

Additive and Qubit Codes from Generalized Toeplitz Construction

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Abstract of Presentation: In a ground-breaking paper, Calderbanks et al. established that 0-dimensional quantum stabilizer (qubit) codes can be represented by special types of additive subgroups of $GF(4)^n$ called self-dual additive codes. Self-dual additive codes are classified by their length, if they contain a codeword of odd length (Type I) or not (Type II), and their minimum distance (a quantity proportional to the number of errors the code can correct). Self-dual additive codes can be generated by graphs, though many such current codes in literature are constructed from circulant graphs and their variations. We present a new code-construction from generalized Toeplitz graphs to generate new 0-dimensional qubit codes. Let G be a finite group. A generalized Toeplitz graph $\Gamma = T_G(S)$ has the vertex set $V(\Gamma) = G$ and the edge set $E(\Gamma) = \{(v, sv) : s \in S, sv \in G\}$. Building from previous literature, we establish conditions for when a self-dual code generated from a generalized Toeplitz graph is Type I or Type II. Using the generalized Toeplitz construction and randomized computer search, we obtained 55- and 56- length qubit codes that improve the minimum distance of the current codes by 1, and maximum minimum distance Type I 28-length additive code.

Recursive Enumeration of Subsets of Subgroups of F_n by Length Using Coset Dynamics

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Abstract of Report Talk: Let $\mathcal{L}_k$ denote the subset of F_n, the free group on n generators, containing all elements of length k. Let H be any finite index subgroup of F_n, yielding X, the set containing all coset representatives of H in F_n. We will present a method to derive the recurrence relation for H, having the form:

$$|H \cap \mathcal{L}_k| = \sum_{i=1}^{m} \sum_{x \in X} c_{x,k-i} |xH \cap \mathcal{L}_{k-i}|$$

where $m \in \mathbb{N}$ is independent of k and $c_{x,i} \in \{0,\ 1,\ \ldots,\ 2n-1\}$ (where n is the rank of F_n). We adopt the convention that if $i < 0$, then $\mathcal{L}_i = \emptyset$. One interesting application is exploring how we may arrive at how $\frac{|H \cap \mathcal{L}_k|}{|\mathcal{L}_k|}$ tends to $\frac{1}{[F_n:H]}$ as $k \to \infty$.
In this talk, we will describe an algorithm to arrive at the recurrence relation, needing only basic ideas from group theory.

Expansivity on Julia Sets

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Abstract of Report Talk: The expansivity of a rational map on its Julia set is closely related to an invariant quantity known as the asymptotic $\overline{E}^{\infty}$ energy. Using metric graphs embedded in Julia sets, we outline methods developed for bounding and calculating $\overline{E}^{\infty}$ for quadratic and cubic post-critically finite hyperbolic rational maps and demonstrate this quantity's relation to expansivity. Upper bounds are calculated by constructing a map forcing efficient lifts, while lower bounds are calculated through assigning a Markov partition to invariant graphs.

Reconstruction of radiating point sources from single frequency data in a 2D acoustic waveguide

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Abstract of Report Talk: This study is concerned with the inverse source problem for the Helmholtz equation using boundary measurements of the radiated wave u at a fixed frequency in a 2D acoustic waveguide. The inverse source problem is motivated by several applications in physics and engineering, including radar, sonar, and nondestructive testing. We introduce an imaging function to determine the number of radiating point sources as well as their locations and intensities. Our method is justified using Green's representation theorem. Furthermore, we provide a stability estimate for the imaging function and establish a uniqueness result for multiple sources using Holmgren's theorem. Numerical examples demonstrate the efficiency of our method and its robustness against noise in the data.

Representation stability of Vertical Configuration Spaces

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Abstract of Report Talk: Given a family of topological spaces or algebraic structures $\{X_n\}_{n=1}^{\infty}$, one often wants to ask the following question: Does their (co)homology group stabilizes as $n \to \infty$? If so, we can just compute a small number of (co)homology groups and infer the rest through this stability! Such a phenomenon is called homological stability, and appears in many areas of mathematics. However, such stability is sometimes too good to hope for, leading Church, Ellenberg, and Farb to propose a more nuanced concept called representation stability in the last decade. In this framework, although the (co)homology groups may not stabilize as groups, they exhibit stability "up to actions by symmetric groups S_n."
An interesting application of this theory is in the study of vertical configuration spaces. Vertical configuration spaces are topological spaces that generalize the configuration spaces of particles in Euclidean space; these classical configuration spaces have an extensive literature that touches many different subfields of mathematics. In 2021, Bianchi and Kranhold showed that the unordered vertical configuration spaces are homologically stable, while the ordered counterparts are not. We show that when considering the action of the wreath product on the space, the k-th (co)homology groups of the ordered vertical configuration space are representation stable. Furthermore, we use tools from representation theory and classical algebraic topology to provide an alternative proof of the homological stability of unordered vertical configuration spaces.

Hearing Cyclicity from the Spectrum of the Kohn Laplacian on Spherical Space Forms

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Abstract of Report Talk: Although Mark Kac's age-old question "Can one hear the shape of a drum?" has been answered in the negative, it has inspired many others of a similar flavor: how much geometric information about the underlying manifold can one discern, or "hear," from the spectrum of a differential operator? One such class of manifolds are spherical space forms obtained by group action on the $(2n-1)$-sphere in $\mathbb{C}^n$. For a finite free-acting subgroup of the unitary group $\Gamma < U(n)$, this sphere quotient S^{2n-1}/Γ is a CR manifold. In this work, we study the spectrum of the Kohn Laplacian $\Box_b$, a second degree differential operator, on spherical space forms.
Previous work has shown that one can recover the order of the fundamental group Γ from the asymptotics of the spectrum of the Kohn Laplacian. Further, on the 3-sphere S^3 one can hear the exact isomorphism class of the underlying group in $U(2)$ up to conjugation. It is also known that one cannot hear conjugacy class in $U(n)$ for higher dimensional spherical space forms with $n > 3$. We turn our attention to a question with one more degree of generality: hearing cyclicity of the fundamental group in $U(3)$ and $U(4)$. We show that the only finite free-acting subgroups of $U(3)$ are the cyclic subgroups $\mathbb{Z}/n$. This observation clarifies that one can indeed hear cyclicity in $U(3)$. Further, we classify the finite free-acting subgroups of $U(4)$, and numerical calculations on S^7 suggest that one can hear conjugacy class in $SU(4)$.

Chromatic Polynomials for 2-Edge-Colored Graphs

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Abstract of Report Talk: The chromatic polynomial is an important tool in the study of algebraic graph theory, initially introduced in 1912 by Birkhoff to study the four color problem, which counts the number of graph colorings as a function of the number of colors. In 2020, Beatona, Cox, Duffy, and Zolkavich introduced a generalization of the chromatic polynomial to 2-edge colored. We have obtained proofs for multiple facts about this polynomial including a modified method of deletion contraction and a generalization of Whitney's Broken Circuits Theorem that gives a combinatorial meaning to the coefficients of this polynomial. We also investigate the properties of the chromatic polynomial for 2-edge colored graphs and compare them to the ones of the polynomial for a traditional graph.