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Coalescing Ballistic Annihilation

Kimberly M Affeld (affeldkimberly@gmail.com)
Christian M Dean (deanmchris@gmail.com)
Connor Panish (wilsonc1@ufl.edu)
Baruch College [Mentor:Matthew Junge]

Abstract of Report Talk: During the 1980's, physicists introduced a model called ballistic annihilation that mimics chemical reactions. Particles move throughout the real line at fixed velocities and annihilate upon collision. We generalize a recent breakthrough from Haslegrave, Sidoravicius, and Tournier to a variant in which particles sometimes survive collisions. Specifically, we characterize the initial conditions for all particles to be annihilated. Our arguments make use of recursion and hidden symmetries within infinite sums.

Exploring the Berezin Range of Operators on Subspaces of the Hardy Space

Rachel C. Andreola (rachelandreola@gmail.com)
Morgan A. Lucas (mal753@nau.edu)
Caroline Norman (cnorman@bates.edu)
University of Michigan - Dearborn [Mentor:John Clifford]

Abstract of Report Talk: For a bounded linear operator T on a complex Hilbert space H, the numerical range of T, $W(T)$, is given by $W(T) = \{\langle Tf, f, \rangle : \ ||f|| = 1\}$. $W(T)$ is unitarily invariant, and when H is finite-dimensional, $W(T)$ is a compact, convex subset of H that contains the spectrum $\sigma(T)$. Furthermore, when H is two dimensional, $W(T)$ is a possibly degenerate elliptical disk.

When $H = F(X)$ is a reproducing kernel Hilbert space (RKHS) with normalized reproducing kernels k_p, $p \in X$, we can consider the Berezin range of T, which is subset of the numerical range of T given by

$$B(T) = \{\langle Tk_p, k_p\rangle : \ p \in X\}.$$

By contrast, the Berezin range is usually not unitarily invariant, and for $H = \mathbb{C}^n$ with the standard RKHS structure, $B(T) = \operatorname{diag}(T)$, which is only convex if T has constant diagonal. Moreover, in this setting, $W(T)$ cannot in general be expressed as even a countable union of Berezin ranges of unitary conjugates of T.

We examine the Berezin range of linear operators on the RKHS M of polynomials with degree $n-1$ or less on the Hardy Space of the unit disk. Given an orthonormal basis $\{f_1, f_2, ..., f_n\}$ on M, the reproducing kernel is computed by $K_p^M(z) = \sum_{j=0}^{n} f_n(z)\overline{f_n(p)}$.

When the dimension of M is two, we show that the numerical range of T can be covered by the union of three or fewer Berezin ranges of unitarily similar operators, and that we may use two or less if and only if T is normal. In addition, for an n-dimensional subspace M of H^2 we determine the conditions on a normal operator T acting on M for which the number of Berezin ranges that cover the numerical range is one, two, or uncountable.

Applications of Biases in Second Moments of Elliptic Curves

Zoe X. Batterman (zxba2020@mymail.pomona.edu)
Aditya Jambhale (aj644@cam.ac.uk)
Williams College [Mentor:Steven J. Miller]

Abstract of Report Talk: Elliptic curves (E: $y^2 = x^3 + ax + b$ with a, b integers) are one of the most important objects in number theory, playing a crucial role in Wiles' proof of Fermat's Last Theorem. They have an associated L-function whose coefficients arise from counting solutions mod p, and their order of vanishing at the central point is conjecturally equal to the group of rational solutions; this is the famous Birch-Swinnerton-Dyer conjecture, one of the Clay Millennium Problems.
We consider a family of curves $y^2 = x^3 + a(T)x + B(T)$ where $a(T), b(T) \in \mathbb{Z}[T]$, and set $A_r(p) = \sum_{t=0}^{p-1} a_t(p)^r$ (if we divide by p this is the r^{th} moment). Under mild assumptions, Michel proved that the second moment $A_2(p)$ is $p^2 + O(p^{3/2})$; in his thesis, Miller showed that the lower order terms in $A_2(p)$ are related to the behavior of zeros of elliptic curve L-functions, and explained some of the differences with the limiting behavior as the conductors tend to infinity. In his and subsequent work, every family discovered had an interesting property: the largest lower-order term in the second moment expansion that does not average to 0 is on average negative. This is now known as the bias conjecture. No counter-examples were known, as it is extremely difficult to find closed-form expressions for the second moment; to date the conjecture has been verified only for a limited number of families whose Legendre sums are computable. Thus there is a danger that these families are special and do not describe a generic family.
We constructed a family, $y^2 = x^3 + x + T^3$, whose second moment is $p^2 + p$ for primes that are 2 modulo 3 and thus does not follow the bias conjecture. Interestingly, the lower order terms at the primes 1 modulo 3 might be negative enough so that a modified bias conjecture, where we average over the primes, is true. The analysis requires numerous results on Legendre and Gauss sums, as well as converting these curves to nonic surfaces and using powerful methods from algebraic geometry to count the number of points, dealing with issues arising from singular fibers at infinity.

Solving Systems of Polynomials via Analytic Continuation and Monodromy

Timothy Cheek (timcheek@umich.edu)
Georgia Tech [Mentor:Anton Leykin]

Abstract of Report Talk: We invent and implement a new algorithm to find solutions of a generic system of multivariate polynomials with parametric coefficients. The novel approach we employ is analytically continuing through a random set of points in an extended space—this addresses the main shortcoming of previous solvers, which is numerical instability of path trackers near branch points. We rigorously justify the correctness and completeness of our algorithm by drawing from analytic continuation, generalized Maclaurin series (Padé techniques), and the action of the fundamental group. We then demonstrate our algorithm's competitiveness within computational algebraic geometry by testing it on benchmark systems and comparing its performance to state-of-the-art solvers.

Dehn Invariant Zero Tetrahedra

Anas Chentouf (chentouf@mit.edu)
MIT [Mentor:Bjorn Poonen]

Abstract of Report Talk: We contribute previously unknown families of tetrahedra of Dehn invariant zero (DI0), as well as some sporadic tetrahedra. We also show that there are finitely many Dehn invariant zero tetrahedra whose dihedral angles have a five-dimensional span over $\mathbb{Q}$, a first step towards classifying all Dehn invariant zero tetrahedra. We also exhibit a Dehn invariant zero tetrahedron that does not tile, thereby answering a previously open question.

Finite Rank Sequences Over Arbitrary Finite Alphabets

Samuel K Chistolini (skc3@Williams.edu)
Williams College [Mentor:Cesar Silva]

Abstract of Report Talk: In symbolic dynamics, a rank-one sequence is classically defined as an infinite sequence V over the finite alphabet $\Sigma = \{0, 1\}$ such that $v_0 = 0$, and we can build v_{n+1} from v_n, and $\lim_{n\to\infty} v_n = V$. An equivalent condition for rank-1 sequences arises when we consider the collection of all finite words that build a sequence. This is known as the *word-size rank* of a sequence, and there are an infinite number of finite words that build a sequence if and only if the sequence is rank-1. We extend these definitions to their natural analogue over finite rank words, noting that the two definitions do not agree on the rank of many sequences that admit a finite rank construction greater than 1, and in particular, we can construct sequences that admit no rank construction, but do yield an infinite word-size rank construction. We also extend our alphabet to $|\Sigma| = n$ for each $n > 2$ by looking at $\Sigma = \{0, 1, \ldots, n-1\}$, and explore the properties of the dynamical systems associated with sequences in higher alphabets.

Analysis of the Kohn Laplacian and Sublaplacian on Compact Quotients of Quadric Manifolds

Adam Cohen (cadam@reed.edu)
Yash Rastogi (yrastogi@uchicago.edu)
University of Michigan Dearborn [Mentor: Yunus Zeytuncu]

Abstract of Report Talk: Elliptic operators such as the Laplace-Beltrami operator Δ have been thoroughly studied, leading to famous results such as the Atiyah-Singer index theorem, and asymptotics for the eigenvalue counting function on compact manifolds. Much less is known about non-elliptic operators. In this talk we present our results on the study of two non-elliptic operators, the Kohn Laplacian $\Box_b$, and the Sublaplacian, Δ_b, on compact quotients of quadric groups, a prototype for CR manifolds.
Much of the study of these operators has been restricted to strongly pseudoconvex CR manifolds. In this case, the Heisenberg group with the standard structure is the prototypical CR manifold. Quadric groups are a generalization of the Heisenberg group and are prototypes for other CR manifolds. An analogue of Weyl's law, that is, relating the asymptotic growth rate of the eigenvalue counting function to the volume of the manifold, for $\Box_b$ on compact quotients of the strongly pseudoconvex Heisenberg group was discovered on (p, q)-forms by Fan, Kim, and Zeytuncu. Fan later computed Sobolev estimates for $\Box_b$, showing that it is subelliptic on (p, q)-forms with $q = 0, n$, outside of the range of J. J. Kohn's initial proof.
We prove similar results on compact quotients of quadric groups with maximal CR dimension. In particular we obtain asymptotics in a Weyl's law form for the eigenvalue counting functions of $\Box_b$ and Δ_b on certain spaces of (p, q)-forms, and show that it fails to be subelliptic in the remaining cases, with infinite dimensional eigenspaces for arbitrarily large eigenvalues.

On the Size and Complexity of Scrambles

Steven Sofos DiSilvio (ssd2165@columbia.edu)
Krish Y Singal (kys2117@columbia.edu)
Williams College [Mentor:Ralph Morrison]

Abstract of Report Talk: In the setting of chip-firing games on graphs, the gonality of a graph provides a discrete analog of the gonality of an algebraic curve. Given a graph G on n vertices, the strongest known lower bound on its gonality is its scramble number, denoted $\mathrm{sn}(G)$, which generalizes the notion of bramble number by not requiring subgraphs to touch. Echavarria et al. showed in 2021 that $\mathrm{sn}(G)$ is NP-hard to compute, but it is not known to be in NP. While a scramble is a natural certificate candidate, its size is potentially exponential in n. To study this certificate, we introduce the carton number of a graph, defined as the minimum size of a maximum order scramble.

Using $\mathrm{tw}(G)$ as a lower bound on $\mathrm{sn}(G)$, we adapt a result on bramble size by Grohe and Marx and prove via a probabilistic argument that bounded degree graphs on n vertices with scramble number $\Omega(n^{1/2+\epsilon})$ have carton number exponential in n. This invalidates scrambles as NP certificates. Furthermore, it shows that minor-closed families of graphs with bounded degree have scramble number $O(\sqrt{n})$. We also prove via contradiction that carton number is at least $3\mathrm{sn}(G) - n$ for graphs of maximum degree less than $\mathrm{sn}(G)$. Lastly, we study an application of Courcelle's theorem to prove the fixed-parameter tractability of scramble number and its variants.

Completions of Quasi-Excellent Integral Domains

Ammar H Eltigani (ahe2@williams.edu)
David Baron (jdf5@williams.edu)
Williams College [Mentor:Susan Loepp]

Abstract of Report Talk: We use the concept of completions from analysis to study the algebraic structure of quasi-excellent integral domains. These rings are particularly relevant to algebraic geometry as they are conjectured to be the base rings for which the problem of singularities can be resolved. We use the M-adic metric, a metric defined on a local ring R with respect to its maximal ideal M, to define the completion of R. Complete local rings are well understood due to a powerful theorem of Cohen's. Therefore, characterizing the relationship between a local ring and its completion allows us to deduce results about the original ring. Although the completion T of a ring R is known to have some of the same algebraic properties as R, certain algebraic properties are not necessarily shared by T and R. For example, the completion of a local integral domain is not necessarily an integral domain. In fact, in 1986, Lech characterized exactly when a complete local ring is the completion of an integral domain. It is worth noting that Lech's conditions are so weak that just about every complete local ring is the completion of an integral domain.

We ask a similar question about quasi-excellent integral domains and their completions. One might expect that the conditions for a complete local ring to be the completion of a quasi-excellent integral domain would be much stronger than Lech's conditions. We show that this is not the case by proving that T, a complete local ring of characteristic 0 with maximal ideal M, is the completion of a local quasi-excellent integral domain if and only if no nonzero integer of T is a zero-divisor and T is reduced. Furthermore, we prove that T is in fact the completion of a countable local quasi-excellent domain if and only if T is reduced and T/M is countable. Proving that the conditions are necessary is fairly straightforward, however, proving they are sufficient is difficult. To do this, we construct an infinite chain of subrings of T by carefully adjoining transcendental and algebraic elements to a base ring while ensuring that the union of the chain satisfies our theorem's conditions. One notable application of our results is that there is no bound on how non-catenary a quasi-excellent integral domain can be, a result that was previously unknown.

Distance graph embedding problems over finite fields

Xinyu Fang (fxinyu@umich.edu)
Maxwell R Sun (mrsun@mit.edu)
Williams College [Mentor:Alex Iosevich]

Abstract of Report Talk: The Erdős distance problem aims to quantify the extent to which points in large sets of a Euclidean space must determine many distances. More generally, the related problem of locating point configurations with prescribed metric structure within subsets of the Euclidean spaces of large Hausdorff dimension is also widely studied. For instance, Bennett, Iosevich and Taylor proved the existence of arbitrarily long chains in compact subsets $A \subseteq \mathbb{R}^d$ with Hausdorff dimension larger than $\frac{d+1}{2}$. Other progress in this direction involves the existence of cycles and more general distance graph structures.
We consider the finite field analogue of point configuration embedding problems of this style. The finite fields setting is interesting due to number theoretic obstructions that do not arise in the Euclidean setting. Moreover, they provide insight into the latter. Previous work obtained asymptotic counts on the number of embeddings of special families of distance graphs into large subsets of $\mathbb{F}_d^q$, such as paths and cycles, and the count of a general graph in such subsets based on its maximal vertex degree.
We significantly generalize such results by allowing the "distance" between points to be defined more generally. We prove the same results hold when the distance between two points $x, y \in \mathbb{F}_q^d$ is defined by any non-degenerate bilinear or quadratic form. In addition, we obtain results on embedding new families of distance graphs which are formed from certain basic configurations. These include the k-fold Hölder extension of 3-chains and chains of equilateral triangles, which have not been studied previously. Our results in these cases make significant improvements on the bounds given by the general case by exploiting symmetries within the graphs and sophisticated inductive arguments. Moreover, in order to lower bound nondegenerate embeddings, we make clever counting arguments of degeneracies that use the geometry of finite field vector spaces.

Random discrete decomposition and Benford's law

Xinyu Fang (fxinyu@umich.edu)
Maxwell R Sun (mrsun@mit.edu)
Williams College [Mentor:Steven Milller]

Abstract of Report Talk: Benford's Law is a statement about the probability of finding certain leading digits, and describes many theoretical and real-life data sets. Precisely, if the proportion of values with leading digit k is proportional to $\log_{10}(1 + 1/k)$ for all k, then the dataset is said to exhibit weak Benford behavior; if, moreover, the logarithms of the values modulo 1 are equidistributed, then the set exhibits strong Benford behavior. This phenomenon has importance in many fields, including applied mathematics, accounting, number theory, physics and fraud detection.

Inspired by particle decay, the Benfordness of a discrete stick fragmentation process was studied by Becker et al. For a stick of integer length L, the stick is cut at a random integer point from 1 to $L - 1$. This is perpetually repeated on the left segments until we have a stick of length 1. They showed that, as $L \to \infty$, the resulting collection of stick lengths exhibits strong Benford behavior. They conjectured that Benfordness would also result if the "stopping lengths" are allowed to take values from certain nice sequences of integers.

We prove their conjecture for the case when the breaking stops at sticks of even integer or unit length. This is an important variation of the original problem because it corresponds to the physical process of particle decay that terminates at certain stable isotopes of given nuclear sizes. Our proof involves approximating the discrete process with a continuous one, estimating the error and applying results from the known continuous case. We further conjecture, based on experimental data, and expect to prove that similar results hold for breaking that stops at other special sequences, such as certain residue classes modulo various integers. Notably, we find that if we consider the aggregated distribution of the resulting stick lengths from many trials, then we get Benfordness from a much wider family of fragmentation processes.

The VC Dimension of the Heisenberg Group

Kayta J Gheorghian (gheorghk@bc.edu)
The Ohio State University [Mentor:Caroline Terry]

Abstract of Report Talk: Let X be a set and F a family of subsets of X. Then for any subset $A \subseteq X$, we say that F shatters A if for each $A' \subseteq A$ there exists $F \in F$ such that $F \cap A = A'$. The Vapnik-Chervonenkis (VC) dimension of X with respect to F is

$$\max_{A \subseteq F} |A| : F \text{ shatters } A.$$

VC-dimension first arose in machine learning and is a measure of complexity of a set system. A canonical example of a set system with bounded VC-dimension is that of axis-parallel boxes in $\mathbb{Z}^3$. This set system has VC-dimension 6, and can be viewed as the set of translates of certain balls the word metric in the group $\mathbb{Z}^3$. In this talk we present work on a similarly defined set system, this time in the Heisenberg group. In particular, we define the Heisenberg be defined as

$$H(\mathbb{Z}) := \left\{ \begin{matrix} 1 & a & b \\ 0 & 1 & c : a, b, c \in \mathbb{Z} \\ 0 & 0 & 1 \end{matrix} \right\}.$$

$H(\mathbb{Z})$ forms a group under matrix multiplication. We define a ball centered at the origin as

$$B_{r_1,r_2,r_3} = \left\{ \begin{matrix} 1 & a & b \\ 0 & 1 & c : |a| \leq r_1, |b| \leq r_2, |c| \leq r_3; r_1, r_2, r_3 \in \mathbb{Z}_{\geq 0} \\ 0 & 0 & 1 \end{matrix} \right\}$$

and construct the family F as the collection of all left translates of all such balls. We show the VC-dimension of F is 6.

Option Pricing under Stochastic Volatility, Change in Equity Premium, and Interest Rates in a Complete Market

Nicole Hao (yh397@cornell.edu)
Echo Li (li.11789@buckeyemail.osu.edu)
Diep Luong Le (ngocdiepluong256@gmail.com)
Ohio State University [Mentor:John Holmes]

Abstract of Report Talk: In 1973, economists Fischer Black, Robert Merton, and Myron Scholes published the Black-Scholes-Merton (BSM) model that revolutionized options pricing. The BSM PDE is used to calculate the fair price of options (financial derivatives that give the holder the right to buy or sell an underlying asset within a specific timeframe). They assume the stock price is described by geometric Brownian motion

$$dS(t) = (\mu + X + R)Sdt + \sigma_S S dW_1(t),$$

where, in their model, μ, X, R and σ_S are constant. Hence, they assume that the time-varying factors such as changes in the risk-free interest rate, equity premium, and volatility of the underlying stock price are constant.

Since then, several authors have constructed models which address some of these limitations. The Cox-Ingersoll-Ross (CIR) model effectively describes the dynamic behavior of interest rates, and can be written as

$$dR(t) = \kappa_R(r - R(t))dt + \xi(\rho_R dW_1(t) + \sqrt{1-\rho_R^2}dW_4(t)).$$

Heston's time-varying variance model (1993), accounts for changes in the variance of stock prices over time. The variance is described by

$$d\sigma_S(t) = \kappa_S(\sigma - \sigma_S(t))dt + \eta\sqrt{\sigma_S(t)}(\rho_S dW_1(t) + \sqrt{1-\rho_S^2}dW_3(t)).$$

Campbell and Viceira (1999, 2002) incorporated a time-varying equity premium into the model, which captures the fluctuations in the equity premium of stocks. The equation of motion for the equity premium was written as

$$dX(t) = -\kappa_X X(t)dt + \sigma_X(\rho_X dW_1(t) + \sqrt{1-\rho_X^2}dW_2(t)).$$

We revisit these models and present a novel approach to the option pricing problem. In particular, we derive a partial differential equation for the option price based on these three models. Then, we use the finite-difference method to estimate the solutions for the PDE we derived based on initial and boundary conditions given by the particular options we consider. Our techniques are widely applicable to both vanilla and exotic options, and we provide a comprehensive approach to determining the prices of complex financial instruments by addressing the limitations observed in the existing literature. Our findings contribute to the advancement of financial modeling and provide valuable insights for risk management, asset pricing, and investment decision-making in diverse financial contexts.

Power Sum Elements of the G2 Skein Algebra

Alaina Hogan (hoganal@mail.gvsu.edu)
Bodie Beaumont-Gould (lc22-0662@lclark.edu)
Michigan State University [Mentor:Vijay Higgins]

Abstract of Report Talk: Given two knots on a surface, we can multiply them by overlaying their respective diagrams. Extending this operation linearly and applying rules for reducing a knot into a linear combination of simpler diagrams creates an algebraic structure called a skein algebra. This skein algebra is a non-commutative algebra accompanied by a parameter q, and is related to quantum groups. We are interested in studying the center of the skein algebra associated to the exceptional Lie group G_2. Our work extends results from Bonahon-Wong and Lê, who found central elements of the skein algebra for the easier case of SL_2.
To do this, we establish relationships between elements of the G_2 skein algebra and the traces of G_2 matrices. These traces can be described by power sum polynomials. In our main result, we find that the elements corresponding to nth degree power sum polynomials are central when q is a $2n$th root of unity.

Binomial Sets Under $\mathbb{Z}$-Linear Forms

Ryan Jeong (rsjeong@sas.upenn.edu)
Williams College [Mentor:Steven Miller]

Abstract of Report Talk: Some of the most fundamental objects in additive combinatorics are sum and difference sets of subsets of non-negative integers, given by

$$A + A := \{a_1 + a_2 : a_1, a_2 \in A\}, \qquad A - A := \{a_1 - a_2 : a_1, a_2 \in A\}.$$

These sets lie at the core of many of the most important problems in number theory. For example, if P is the set of primes and A_k the set of k^{th} powers of positive integers, then Goldbach's Conjecture, the Twin Primes Conjecture, and Fermat's Last Theorem are respectively the statements that $P + P$ contains all even numbers that are at least 4, that 2 can be generated in $P - P$ via infinitely many representations, and that $(A_k + A_k) \cap A_k = \emptyset$ for any integer $k \geq 3$.

We generalize in the following way: given an integer $h \geq 2$, and a function $L : \mathbb{Z}^h \to \mathbb{Z}$ defined by

$$L(x_1, \dots, x_h) := u_1 x_1 + \cdots + u_h x_h, \quad u_i \in \mathbb{Z}_{\neq 0} \text{ for all } i \in [h],$$

we study, as N grows, the cardinality of

$$L(A) = \{u_1 a_1 + \cdots + u_h a_h : a_i \in A\}$$

and its complement when generating A from a binomial model, i.e., including elements from $\{0, 1, \dots, N\}$ independently in A with probability $p(N)$. Interestingly, depending on the density of the set A, this leads to very different behavior. We identify $N^{-\frac{h-1}{h}}$ as a threshold function. As $p(N)$ goes from below the threshold to above the threshold, then $L(A)$ goes from having almost all sums generated in $L(A)$ being almost surely distinct to having almost all possible sums generated in $L(A)$.

We derive asymptotic formulae for $|L(A)|$ in these different regimes. This generalizes work of Hegarty and Miller in 2008, completely settling two parts of a conjecture of theirs and displacing a third with the correct statement. Our methods rely on recent results on partition asymptotics due to Stanley and Zanello, a classic variant of the Stein-Chen method due to Arratia, Goldstein, and Gordon, and the martingale concentration machinery of Vu.

Braidings on Non-Split Tambara-Yamagami Categories over the Reals

Yoyo Jiang (yjiang70@jhu.edu)
The Ohio State University [Mentor:David Penneys]

Abstract of Report Talk: In 1998, Tambara and Yamagami investigated fusion categories with a single non-invertible simple object and a straightforward set of fusion rules resulting from self-duality. They classified all possible associators on these categories, thereby classifying all monoidal structures. Two years later, Siehler classified all braiding structures on the same set of fusion rules. This project investigates braidings on a generalization of these fusion rules to a setting where simple objects are no longer required to be split. In particular, we classified braidings on fusion categories over the reals using techniques from the recent paper by Plavnik, Sanford and Sconce that classifies associators on these non-split categories, considering the three possible cases where objects are real, complex or quaternionic. In this talk, we will introduce some key techniques used in our project that allow us to perform graphical computations with string diagrams, and we will demonstrate some examples of these computations before discussing the results.

Coalescing Results for the Distance Matrix

Jiah Jin (jiah.jin@cooper.edu)
Said Bin Mahamud (sajidbmahamud@reed.edu)
Iowa State University [Mentor:Steve Butler]

Abstract of Poster Presentation: Given a graph G, the distance matrix $D(G)$ has rows and columns indexed by the vertices of G with the (u, v) entry being $\text{dist}(u, v)$, the distance between vertices u and v. Let $G \overset{B}{\circ} H$ be the graph resulting from gluing a copy of H onto each vertex of $B \subseteq V(G)$ in G. We say (G_1, B_1) and (G_2, B_2) are gluing cospectral if $G_1 \overset{B_1}{\circ} H$ and $G_2 \overset{B_2}{\circ} H$ are distance cospectral (have the same eigenvalues for the distance matrix) for all graphs H. We give a sufficient condition for (G_1, B_1) and (G_2, B_2) to be gluing cospectral in terms of a block similarity matrix that commutes with the all ones matrix. This sufficient condition explains all gluing cospectral pairs up through 9 vertices. Moreover, when this condition holds, if (G_1, B_1) and (G_2, B_2) are gluing cospectral, then $(G_1, V(G_1) \setminus B_1)$ and $(G_2, V(G_2) \setminus B_2)$ are also gluing cospectral.

Digitally Restricted Ostrowski Expansions

Khaiylah Johnson Bustamante (kjohnsonbustamante@mercy.edu)
Ellie Lew (elew@brynmawr.edu)
Yale University [Mentor:Felipe A. Ramirez]

Abstract of Report Talk: The standard middle-thirds Cantor set is the set of $x \in [0,1]$ having base-3 expansions which do not contain the digit 1. The Hausdorff dimension of the Cantor Set is $\log 2/\log 3$. In fact, any Cantor-like set with digital restrictions in base $b > 1$ has Hausdorff dimension $\log a/\log b$, where a is the number of allowed digits. We will discuss recent work where we determine the Hausdorff dimensions of analogously defined sets. Our sets consist of real numbers whose "Ostrowski digits" have been restricted.
Fix an irrational $\beta \in (0,1)$ with continued fraction $\beta = [0; b_1, b_2, \dots]$, and denote $D_n = q_n\beta - p_n$ where p_n/q_n are the convergents of β. The Ostrowski expansion of a real number x with respect to β is an expression of the form

$$x = \sum c_{n+1} D_n$$

subject to the condition that if $c_{n+1} = b_{n+1}$, then $c_n = 0$.
In the sets we investigate, there are restrictions on the digits c_n appearing in the Ostrowski expansions of real numbers x with respect to a fixed β. In the simplest case where $\beta = [0; b, b, \dots]$ we find that our digitally restricted sets have dimension $-\log\alpha/\log\beta$, where α is a real number which is naturally related to the number of allowed Ostrowski digits. On the other hand, if the coefficients of β's continued fraction are allowed to grow, then the digitally restricted sets have a dimension which depends on that growth. We also study "fractal percolation" in this setting, where the digital restrictions are determined by a random process.
Our results have applications in metric Diophantine approximation.

Stiffness matrices and d-dimensional algebraic connectivity

Yunseong Jung (yunseonj@andrew.cmu.edu)
Carnegie Mellon University [Mentor:Alan Lew]

Abstract of Report Talk: A graph G embedded in d-dimensional space is called rigid if there is no continuous motion of the vertices that preserves the distance between all pairs of adjacent vertices, except for trivial motions. Jordán and Tanigawa recently introduced the notion of d-dimensional algebraic connectivity $\alpha_d(G)$, which is a quantitative measure of the d-dimensional rigidity of G, defined in terms of the eigenvalues of certain "stiffness matrices" associated to different embeddings of G. The question of determining the value of $\alpha_d(G)$ turns out to be surprisingly difficult, and an exact solution is not known even for some of the simplest examples.
In this talk, we will describe a special structure of the eigenvectors of the stiffness matrix when the embedding has "repeated points" (that is, when several vertices of G are mapped to the same point in space) and derive a method to simplify the computation of the stiffness matrix eigenvalues. Using this, we prove new lower bounds on the 2-dimensional algebraic connectivity of complete bipartite graphs, and provide an alternate, simplified proof for the d-dimensional algebraic connectivity of generalized stars.

Limiting Behavior in Missing Sums of Sumsets

Rauan Kaldybayev (rk19@williams.edu)
Christopher H Yao (chris.yao@yale.edu)
Williams College [Mentor:Steven Miller]

Abstract of Report Talk: Many of the most important problems in number theory concern the sum or difference set of a given set of integers. For example, Goldbach's conjecture states that if P is the set of all primes, $P + P$ includes all even numbers starting from four; the Twin Prime Conjecture states that two is represented infinitely often in $P - P$; and Fermat's Last Theorem states that if S_k is the set of k-th powers of positive integers, the intersection of $S_k + S_k$ with S_k is empty whenever $k > 2$. We report on recent results about the distribution of the number of missing summands in $A + A$, where $A \subseteq \mathbb{Z}_{\geq 0}$ is chosen randomly such that for every n, the probability of $n \in A$ is equal to p.
Our main result is an exponential upper bound $O\left(m(1-p^2)^{m/2}\right)$ on the probability of missing m or more summands, which is significantly tighter than the previously known inverse-polynomial bound $O(1/m^2)$ resulting from Chebyshev's inequality. To prove tightness, we give an exponential *lower* bound $\mathbb{P}\left(|(A+A)^c| \geq m\right) = \Omega\left((1-p)^{m/2}\right)$ that is close to our upper bound. We also derive an exact formula for the probability of missing one or two given summands. We show that the expectation value of $|(A + A)^c|$ is exactly $2/p^2 - 1/p - 1$ and calculate the variance of $|(A+A)^c|$ to high precision. We study the probability of missing zero summands, $\mathbb{P}\left(A + A = \mathbb{Z}_{\geq 0}\right)$, which turns out to be a non-analytic function of p. Our proof proceeds by considering infinite sets A; to connect our work with previous literature, which only considered finite A, we reduce the finite case to the infinite case. Time permitting, we will discuss extensions of our results to A^{+k}, the k-fold sum of A with itself, and $A^{+\infty}$, the set of all numbers expressible as an arbitrary sum of elements of A.

The Failed Zero Forcing Number of a Graph

Chirag Kaudan (ckaudan02@gmaill.com)
Rachel M Taylor (13milan17@gmail.com)
Rochester Institute of Technology [Mentor:Darren Narayan]

Abstract of Report Talk: Given a graph G, the zero forcing number of G, is the smallest cardinality of any set S of vertices on which repeated applications of the forcing rule results in all vertices being in S. The forcing rule is: if a vertex v is in S, and exactly one neighbor u of v is not in S, then u is added to S in the next iteration. In 2015, Fetcie et al. defined the failed zero forcing number of a graph which is the cardinality of the largest set of vertices which fails to force all vertices in the graph. They posed the question of characterizing all graphs with a failed zero forcing number of 2. In 2021, Gomez et al. provided a solution using a brute force approach. Using novel methods, we obtained a streamlined proof that there are exactly 15 graphs with a failed zero forcing number of size 2. Furthermore, we discovered that there are exactly 68 graphs with a failed zero forcing number of 3, and exactly 662 graphs graphs with a failed zero forcing number of 4.

Laplacian Eigenmaps and Orthogonal Polynomials

Jonathan A. Kerby-White (jkerbywh@iu.edu)
Yiheng Su (ysu24@colby.edu)
University of Connecticut [Mentor:Luke Rogers]

Abstract of Poster Presentation: In this poster, we investigate the nature of eigenmaps of graph Laplacians, in particular approximating the eigenmaps using Chebyshev polynomials with controlled errors. We begin by defining several types of Laplacian operators on graphs whose vertices only connect to their nearest neighbors. We define eigen-coordinates $(f_1(x_j), f_\ell(x_j)) \in \mathbb{R}^2$ for $j = 1, ..., n$ where f_ℓ are the eigenfunctions corresponding to the ℓ^{th} smallest nonzero eigenvalues for the Laplacian operators. Next, we derive a general formula for eigenvalues and eigenfunctions for three specific Laplacians: regular, probabilistic, and periodic. We then prove the eigen-coordinates for these operators are exactly Chebyshev polynomials of the first kind $T_\ell(x)$.
Eigenmaps are important in analysis, geometry, and machine learning, in particular, in nonlinear dimension reduction. However, few studies focus on the relationship between eigen-coordinates of graph Laplacians and Chebyshev and other polynomials. To this end, we study the error between eigen-coordinates and Cheyshev polynomials, $error(x) = |f_\ell(x) - T_\ell(f_1(x))|$. In our next step, we will consider constructing graph Laplacian operators on $[-1, 1]^2$, the sphere S^2, and the Sierpinski gasket. We would also like to explore these operators when the points are selected randomly with various distributions, such as uniform and Gaussian. We will compare the eigen-coordinates of graph Laplacian operators constructed from those intervals with families of orthogonal polynomials, such as the Hermite polynomials and possibly exceptional orthogonal polynomials.

Generalizations of the Erdos–Ginzburg–Ziv theorem via topology

Jacob Lehmann Duke (jl34@williams.edu)
Hannah Park-Kaufmann (hk9622@bard.edu)
Carnegie Mellon University [Mentor:Florian Frick]

Abstract of Report Talk: Erdős, Ginzburg, and Ziv showed that any sequence of $2n-1$ numbers in $\mathbb{Z}/n$ has a subsequence of length n that sums to zero. This zero-sum subsequence may be unique, which is prohibitive to proving constrained or quantitative generalizations of the Erdős–Ginzburg–Ziv theorem. However, for a prime p a sequence of p numbers in $\mathbb{Z}/p$ sums to zero if and only if it is a difference of two permutations of $\mathbb{Z}/p$. We develop a novel topological approach, based on the non-existence of continuous maps that commute with certain symmetries, to prove that the permutations whose difference gives the zero-sum subsequence may be strongly restricted. More specifically, we argue that $\mathbb{Z}/p$-equivariant maps from the chessboard complex $\Delta_{p,2p-1}$ must hit the barycenter of properly chosen codomains, guaranteeing a zero-sum subsequence. We then use this to provide exponential lower bounds (in p) to the number of relevant permutations, thus establishing a quantitative generalization of the Erdős–Ginzburg–Ziv theorem. Our topological approach naturally lends itself to proving fractional strengthenings of the theorem as well, a special case of which is the following extension: For any sequence of $2p-1$ sets $A_1, \ldots, A_{2p-1} \subset \mathbb{Z}/p$, each of cardinality k, there is a subsequence $A_{i_1}, \ldots, A_{i_p}$ such that $A_{i_1}+1, A_{i_2}+2, \ldots, A_{i_p}+p$ is a balanced family of subsets of $\mathbb{Z}/p$, in the sense that these translated sets determine a hypergraph that admits a perfect fractional matching. The $k=1$ case recovers the classical Erdős–Ginzburg–Ziv theorem.

Study of Variable Step Method of Dahlquist, Liniger, and Nevanlinna for the Allen-Cahn Model

Dianlun Luo (dl3572@columbia.edu)
The Ohio State University [Mentor:Yulong Xing]

Abstract of Report Talk: The Allen-Cahn equation serves as a fundamental phase-field model in various scientific disciplines, capturing phenomena such as phase transitions and interface dynamics. In this study, we focus on numerically solving the Allen-Cahn model using the DLN (Dahlquist, Liniger, and Nevanlinna) temporal discretization, a two-step numerical method known for its second-order accuracy and stability for arbitrarily non-uniform time steps. The DLN method is an ideal candidate for time adaptivity due to its fine properties of stability and accuracy. We employ the DLN method for time discretization and finite element space for spatial discretization of the Allen-Cahn equation. Within the DLN framework, different ways to tackle the nonlinear forcing term of the Allen-Cahn equation will be investigated, including the DLN-IMEX algorithm treating the linear part of the source function implicitly and the non-linear part explicitly, and the DLN-SAV algorithm by introducing an additional variable to enhance the energy stability of the numerical method. To further optimize the method, we study an adaptive DLN algorithm based on local truncation error. The performance of the numerical solutions will be evaluated by extensive numerical examples. This study offers valuable insights into efficient and robust numerical methods utilized for phase-field models, particularly emphasizing the effectiveness of the DLN method in addressing the Allen-Cahn model.

A Structure Theorem on Doubling Measures for Arbitrary Bases

Zoe Markman (zmarkma1@swarthmore.edu)
Teresa S Pollard (tsp8810@nyu.edu)
Carnegie Mellon University [Mentor:Theresa Anderson]

Abstract of Report Talk: We define an n-adic interval as any interval of the form $[\frac{k}{n^\alpha}, \frac{k+1}{n^\alpha}]$ for any integers k and α. While breaking up the real numbers into a union of n-adic pieces is a frequently used technique in analysis, there are many unanswered questions pertaining to the behaviour of functions and measures on these systems. A 2022 paper by Anderson and Hu constructs a measure which is p and q-adic doubling for distinct primes p, q but not doubling overall. The construction of this measure relies heavily on an established number theory framework, which allows for the selection of an infinite family of q-adic intervals "close enough" to p-adic intervals such that the Lebesgue measure can be reweighted on these q-adic intervals in a way that preserves p and q-adic doubling, while breaking doubling overall. We generalize the number theory involved in this construction to create a measure that is n-adic doubling and $m_1, \ldots, m_k$-adic doubling for any $n \in \mathbb{N}_{>1}$, such that each m_i is a multiple of some c coprime to n. We then provide a further generalization of the analysis used by Anderson and Hu to construct a measure that is n-adic doubling for every $n \in \mathbb{N}_{>1}$ but not doubling overall. Finally, we present several applications of our results involving the intersection of weight and function classes, including a new result on functions of vanishing mean oscillation.

Quantum chromatic numbers of quantum graph products

Meenakshi McNamara (mcnama20@purdue.edu)
Purdue University [Mentor:Rolando de Santiago]

Abstract of Report Talk: Quantum graphs and quantum chromatic numbers are closely tied to quantum error-checking problems and quantum communication. Quantum graphs are a generalization of graphs using operator algebras and represent quantum relations. Quantum colorings are defined in terms of random strategies for non-local games using entanglement. We will introduce both these concepts and discuss existing bounds on quantum chromatic numbers and our work to expand upon them. We develop a method of defining quantum versions of classical graph products and find bounds for the resulting quantum chromatic numbers of these quantum graph products. In particular, we define a quantum b-fold chromatic number which we use to derive upper bounds for the lexicographic product that are analogous to those in the classical case, and also provides us with a new definition of a fractional version of the quantum chromatic number.

Hausdorff dimensions of inhomogeneously arbitrarily well approximable sets

Jack H Moffatt (jack.moffatt@yale.edu)
Yale University [Mentor:Felipe Ramirez]

Abstract of Report Talk: Given $y \in \mathbb{R}^d$ and a monotonically decreasing function $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$, a vector $x \in \mathbb{R}^d$ is said to be ψ-approximable with inhomogeneous parameter y if infinitely many $q \in \mathbb{N}$ satisfy $\|qx + y\| < \psi(q)$, where $\|\cdot\|$ denotes distance to the nearest integer. It was shown in 2018 that the well approximable vectors—i.e., those for which Dirichlet's theorem can be improved by an arbitrarily small multiplicative factor—can be characterized as the vectors x for which there is at least one inhomogeneous parameter y with which the vector x can be approximated at every monotone divergent rate ψ with $\sum_{q \geq 1} \psi(q)^d = \infty$. By the doubly metric formulation of Khintchine's theorem, the set of inhomogeneously ψ-approximable points for each of these functions is full measure, but this measure drops to 0 over the uncountable intersection. It is natural to ask: what is its Hausdorff dimension?
In one dimension, continued fractions provide a powerful tool for exploring these sets. In particular, we can represent real numbers in their Ostrowski expansion, an enumeration system based on the errors in approximations of successive continued fraction convergents. Our research computes the Hausdorff dimension of the vertical cross-sections of this set by considering fractals contained within them, which are realized as Cantor set-like digital restrictions in these Ostrowski expansions. In higher dimensions, we conjecture that these results generalize analogously, and we explore methods about how these generalizations may arise.

Modeling the vanishing of L-functions at the central point

Akash L Narayanan (anaray@umich.edu)
Zoe X. Batterman (zxba2020@mymail.pomona.edu)
Williams College [Mentor:Steven Miller]

Abstract of Report Talk: While the primes are the building blocks of the integers, their distribution and properties are difficult to directly determine. One of the most successful approaches is through the Riemann zeta function. Defined by $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$ for $\mathrm{Re}(s) > 1$, by the fundamental theorem of arithmetic it equals $\prod_p (1 - 1/p^s)^{-1}$ (where p ranges through the primes). This relation allows us to pass from knowledge of the integers to that of the primes. More generally, one can consider L-functions $L(s, f) := \sum_n a_f(n)/n^s$, where the $a_f(n)$'s arise from number theory problems (examples include Dirichlet characters, related to primes in arithmetic progression, and elliptic curves, related to Fermat's Last Theorem). The Riemann Hypothesis states that all the non-trivial zeros have real part 1/2; the Prime Number Theorem is equivalent to all the zeros of $\zeta(s)$ have real part less than 1.

If all the zeros have real part 1/2, we can look at the distribution of spacings between them. In the 1970s it was observed that many of the statistics here are similar to what one sees when studying energy levels of heavy nuclei, or eigenvalues of random matrices. The Katz-Sarnak philosophy, proved in many restricted cases, states that the distribution of zeros near $s = 1/2$ (the central point) of a family of L-functions as their conductors tend to infinity matches the distribution of eigenvalues near 1 of a random matrix ensemble from one of the classical compact groups as the matrix size goes to infinity. In particular, as the conductor N of a family of elliptic curve L-functions grows, the limiting behavior agrees with orthogonal matrices.

The situation is very different for finite conductors. In 2006, Miller found an unexpected repulsion of low-lying zeros for elliptic curve L-functions of finite conductor. In 2011, Dueñez, Huynh, Keating, Miller, and Snaith created an excised orthogonal ensemble which explained Miller's observation. They noticed that excising matrices whose characteristic polynomials at 1 are larger than some given cut-off value leads to an ensemble of matrices with similar behavior as elliptic curves with finite conductors. The motivation is due to a discretization in the values of elliptic curve L-functions at the central point. We greatly generalize the excised ensemble theory to Hecke eigenforms of general even weight; elliptic curves are weight 2 examples. We find the repulsion decreases significantly as the weight increases, which can be explained using advanced theory on the discretization of central values. Our theoretical predictions are confirmed with extensive computations of quadratic twists of a fixed cuspidal newform. The key ingredient in the proof is incorporating the arithmetic factors, which necessitate computing lower-order terms in statistics such as pair correlation. The numerical calculations are computationally intensive, requiring the construction of rapidly converging series expansions.

Interlacing of Zeros of Period Polynomials

Jingchen Ni (jni33@emory.edu)
Laura L O'Brien (lauraobrien@ucsb.edu)
Clemson University [Mentor:Xue Hui]

Abstract of Report Talk: Period polynomials of modular forms are the generating functions for special values of L-functions, which lead to significant breakthroughs in areas such as the arithmetic of elliptic curves, the Birch-Swinnerton-Dyer conjecture, and the Bloch-Kato conjectures. The paper by Seokho Jin, Wenjun Ma, Ken Ono and Kannan Soundararajan proved that all the zeros of the degree $k-2$ period polynomial $r_f(z)$ associated with newform $f \in S_k(\Gamma_0(N))$ are on the circle $|z| = 1/\sqrt{N}$ and become equidistributed when either k or N is large. Their result is called the "Riemann hypothesis" for period polynomials of modular forms.

In our research, we analyse the sample zeros of $r_f(z)$ given by Jin et.al., to show that zeros of $r_f(z)$ not only are equally distributed, but also interlace with zeros of period polynomial associated with $f' \in S_{k'}(\Gamma_0(N))$, whenever $k' > k \geq 78$ or $N \geq 196476$. When either k or N is large enough, the upper bound of the distance between sample and actual zeros, $\frac{C(K,N)}{2^m\sqrt{N}}$, can be arbitrarily small. And thus the interlacing and Stieltjes interlacing between sample zeros ensure the (Stieltjes) interlacing of actual zeros of $r_f(z)$.

Bounding VC-Dimension in Non-abelian Groups

Tora Ozawa (tozawa@u.rochester.edu)
David Zeng (david.zeng@yale.edu)
Ohio State University [Mentor: Gabriel Conant]

Abstract of Report Talk: In the 1960s and 70s Vladimir Vapnik and Alexey Chervonenkis were developing new ideas in statistical learning theory. They introduced VC-Dimension, a measure of the expressive power of sets. Since then, the notion has been thoroughly applied in areas such as extremal combinatorics, additive combinatorics, group theory, model theory. In particular, lots of exploration has been done in finite fields and $\mathbb{R}^n$. Some elementary examples in $\mathbb{R}^n$ include the VC-dimension of spheres, axis-parallel boxes in $\mathbb{R}^n$. These sets can be thought of as generalizations of arithmetic progressions. However notions of VC-dimension in non-abelian group structures are much less understood.
And so our work has been focused on free groups and nilpotent groups. The main kinds of sets we have been investigating within these groups are (non-abelian) progressions. To bound the VC-dimension of a progression in a free group, we have examined the properties of Cayley graphs and translated the problem to a graph theoritic one about relative distances of end points of a tree. Alternatively, we have formulated the problem of VC-dimension of progressions in the Heisenberg group into one of sets of polynomials. In both cases, we have also used computational evidence to gain insights on the behavior of different sets in these groups. We have also employed well known results in the VC-dimension and model theory literature: the Sauer-Shelah lemma and the model theory of semi-algebraic sets. Consequently, we have two main theorems:

Theorem 1. *The VC-Dimension of progressions in the free group, F_k, on k generators has the following bounds:*

$$k \leq VC_{F_k} \leq 3k - 1 \tag{1}$$

Theorem 2. *The VC-Dimension of progressions in the Heisenberg group with respect to its canonical generators, is bounded by a constant independent of the size of the progressions.*

Properties of Families of Graphs with Forbidden Induced Subgraphs

Christian Pippin (cpippi1@students.towson.edu)
Towson University [Mentor:Vince Guingona]

Abstract of Report Talk: Many families of graphs can be characterized by a set of graphs which are forbidden from appearing, in some particular sense, in any graph in the family. A famous example is the family of planar graphs. Kuratowski's theorem states that planar graphs are exactly the graphs which forbid the complete graph K_5 and the complete bipartite graph $K_{3,3}$ as subdivisions. Our research focuses on families of graphs where particular graphs are forbidden from appearing as induced subgraphs.
We investigate whether these families have certain properties. The first property we study is indivisibility, which is a special case of the Ramsey property. A family $\mathcal{F}$ of graphs is indivisible if for any graph A in $\mathcal{F}$, there exists a larger graph B in $\mathcal{F}$ such that every 2-coloring of B yields a monochromatic copy of A as an induced subgraph of B. We also consider whether these families have the amalgamation property. A family $\mathcal{F}$ of graphs has the amalgamation property if for all graphs A in $\mathcal{F}$ and all embeddings of A into some B_0 and into some B_1 in $\mathcal{F}$, there exists a C in $\mathcal{F}$ where B_0 and B_1 have embeddings into C which agree on the images of A.
We have found that the family $\text{Forb}(P_n)$, which forbids a path on n vertices, has the amalgamation property for $n \leq 3$; however for $n > 3$, amalgamation does not hold. Additionally, we know that $\text{Forb}(P_n)$ is indivisible for $n \leq 4$ and conjecture that $\text{Forb}(P_n)$ is indivisible for $n > 4$. Some of the other families of graphs we are interested in studying are chordal graphs, which forbid cycles of length greater than 3; and perfect graphs, which forbid cyles of odd length greater than or equal to 5 and their complements.

The sequence of ranks of the powers of a sequence

Jesus Omar Sistos Barron (js56758@georgiasouthern.edu)
Baruch College [Mentor:Eric Rowland]

Abstract of Report Talk: Constant-recursive sequences are those in which later terms can be obtained as a linear combination of the previous ones. The rank of a sequence is the minimal number of previous terms required for such recurrence. For a sequence $s(n)_{n\geq 0}$, we study the sequence $(rank(s(n)^{m+1}))_{m\geq 0}$. We prove that, in general, this sequence is eventually polynomial; and for a fixed initial rank, we study the possible polynomials that will eventually describe it. We do this by looking at the roots of the characteristic polynomials and studying these from a pattern-avoidance perspective.

Entropy Formula for Generalized S-gap Shifts

Amy Somers (asomers@ucsb.edu)
Cristian Ramirez (cramirez@berkeley.edu)
Ohio State University [Mentor:Daniel Thompson]

Abstract of Report Talk: Given an $S \subset \mathbb{Z}_{\geq 0}$, an S-gap shift is defined to be the shift space consisting of all sequences in $\{0,1\}^{\mathbb{Z}}$ such that any two 1's are separated by a word 0^n for some $n \in S$. The S-gap shifts have a dynamically and combinatorially rich structure. Dynamical properties of the S-gap shift can be related to the properties of the set S. This interplay is particularly interesting when S is not syndetic such as the example when S is the set of prime numbers or when $S = \{2^n\}$. It is a well known result that the entropy of the S-gap shift is given by $h(X) = \log(\lambda)$, where $\lambda > 0$ is the unique solution to the equation $\sum_{n\in S} \lambda^{-(n+1)} = 1$. Fix a point w of the full shift $\{0,1\}^{\mathbb{Z}}$. We introduce a generalization of the S-gap shift consisting of sequences in $\{0,1,2\}^{\mathbb{Z}}$ in which any two 2's are separated by a subword u of w such that $|u| \in S$. In the case that w is a periodic point, we extend the formula for the entropy of the S-gap shift to a formula describing the entropy of this new class of shift spaces. Additionally we investigate the dynamical properties of this generalization of the S-gap shift.

Characterizing LEF Groups

Russell P Stetson (rps132@scarletmail.rutgers.edu)
Rutgers University - New Brunswick [Mentor:Simon Thomas]

Abstract of Report Talk:
For $n \geq 1$, let S_n be the symmetric group on $\{1, 2, \cdots, n\}$. Then a group G is said to be *locally embeddable into finite groups* if for every finite subset $F \subseteq G$, there exists an injection $\varphi : F \to S_n$ for some $n \geq 1$ such that whenever g, h, $gh \in F$, then $\varphi(gh) = \varphi(g)\varphi(h)$. In this case, we say that G is an LEF group.
In the group theoretic literature, LEF groups are usually characterized in terms of embeddings into ultraproducts of finite symmetric groups. It is natural to ask whether there is a characterization in terms of the more concrete notion of a reduced product of finite symmetric groups. In more detail, let $P = \prod_{n\geq 1} S_n$ be the full direct product and let N be the normal subgroup of elements $(\pi_n) \in P$ such that $\pi_n = 1$ for all but finitely many $n \geq 1$. Then the reduced product is the quotient $P_0 = P/N$.
We have shown that it is neither provable nor disprovable using the classical ZFC axioms of set theory that if G is a group such that $|G| \leq 2^{\aleph_0}$, then G is an LEF group if and only if G embeds into P_0. We are currently working on obtaining the analogous results for sofic groups.

Topological order and boundary algebras for Kitaev's quantum double model

Shuqi Wei (shuqi_wei@berkeley.edu)
The Ohio State University [Mentor:Daniel Wallick]

Abstract of Report Talk: Kitaev's quantum double model is a well-known example of a topologically ordered quantum spin system. Such systems are of interest in physics since the ground state is a quantum error correction code, and they are also modeled by interesting mathematics such as topological quantum field theory and unitary modular tensor categories. Jones, Penneys, Naaijkens, and Wallick provided a description of topological order involving boundary algebras that is a strengthening of the error correction code property. I am adapting methods developed by Naaijkens to prove that the quantum double model satisfies this stronger form of topological order and am computing the boundary algebras for this model.

Bergman kernels of higher-dimensional monomial polyhedra

Mary Wright (wrigh5me@cmich.edu)
Jonathan Gregory (grego3j@cmich.edu)
Central Michigan University [Mentor:Debraj Chakrabarti]

Abstract of Report Talk: For a domain (open connected set) $\Omega \subset \mathbb{C}^n$, the *Bergman kernel* $K_\Omega : \Omega \times \Omega \to \mathbb{C}$ is the reproducing kernel of the *Bergman space*, the space of square integrable holomorphic functions on Ω. In general, it is impossible to find the Bergman kernel of a domain explicitly, except in a few highly symmetric situations like the unit ball. In this talk, we present a closed form formula for the Bergman kernel of a so-called *monomial polyhedron*, i.e a bounded domain of the form

$$\mathcal{U}_B = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^{b_1^j} \dots |z_n|^{b_n^j} < 1, 1 \leq j \leq n\},$$

where B is an $n \times n$ integer matrix. These domains (which include the classical Hartogs triangle as an important special case) have emerged recently as interesting counterexamples in the study of L^p-boundedness of the Bergman projection. Our computation uses Bell's transformation formula for the Bergman kernel under a proper holomorphic mapping and a representation of monomial polyhedra as a quotient domain under the action of an Abelian group of automorphisms. After explaining the computation, we discuss the algebraic properties of the kernel, which turns out to be a rational function. Though several authors have obtained special cases of the formula in the past, this is the first fully general formula for the Bergman kernel of a monomial polyhedron.

Geometry of the Numerical and Berezin Range

Edwin Xie (eax.2004@gmail.com)
UM-Dearborn [Mentor:Alan Wiggins]

Abstract of Report Talk: The numerical range of a bounded linear operator T on a complex Hilbert space H is defined as

$$W(T) = \{\langle Tf, f\rangle : ||f|| = 1\}$$

We present a new perspective on the numerical range of different $n \times n$ matrices as varying "shadows" of an embedding of $\mathbb{C}P^{n-1}$. This framework gives us geometric proofs of the elliptic range theorem and the Toeplitz-Hausdorff theorem.

On the Hardy-Hilbert space $H^2(\mathbb{D})$ of the open unit disk $\mathbb{D}$, the Berezin range is defined as

$$B(T) = \{\langle T\hat{k}_p, \hat{k}_p\rangle : p \in \mathbb{D}\}$$

where $\hat{k}_p$ is the normalized reproducing kernel. While the numerical range is invariant under unitary transformation, and it is convex by Toeplitz-Hausdorff, in general the Berezin range loses both of these properties.

It is natural to ask when the Berezin range is convex: We characterize the convexity of the Berezin range for two-dimensional subspaces of $H^2(\mathbb{D})$. It is also natural to ask how many unitarily similar operators we need to cover a numerical range: We show that uncountably many operators are needed if the boundary of the numerical range is not smooth.

Circle Packings from Tilings of the Plane

David Yang (dyang5@swarthmore.edu)
Swarthmore College [Mentor:Ian Whitehead]

Abstract of Report Talk: We introduce a new class of fractal circle packings in the plane which generalize the polyhedral circle packings defined by Kontorovich and Nakamura. Our packings are constructed by repeatedly reflecting an infinite configuration of base circles across an infinite configuration of dual circles. The existence and uniqueness of these packings are guaranteed by infinite versions of the Koebe-Andreev-Thurston circle packing theorem. We prove structure theorems that give a complete description of the symmetry groups for all such packings. The three main examples of circle packings we study are triangular, square, and hexagonal packings. We study these examples from a number-theoretic point of view, proving integrality properties and quadratic and linear relations for the curvatures of circles in the packing.

On the Existence of Vertex Disjoint Rainbow Triangles in Complete Graphs

Cyrus Jonathan Young (cyrusjy@uci.edu)
Wohua Zhou (wohuazhou@gmail.com)
Carnegie Mellon University [Mentor:Juergen Kritschgau]

Abstract of Report Talk: Rainbow problems ask for conditions of edge-colored simple graphs which are sufficient to guarantee the existence of a particular rainbow subgraph. These problems have been studied extensively in the context of rainbow matchings. Kritschgau [*Elec. J. Combin. 27(3) (2020)*] utilized a greedy algorithm, which can be thought of as an inductive proof containing multiple inductive hypotheses, to determine the appearance of rainbow matchings of a certain size within a graph. Our work extends these ideas to find vertex disjoint rainbow triangles. In this presentation, we will discuss algorithms on complete graphs, which prove that a complete graph with enough colors must contain a specific number of vertex disjoint rainbow triangles.

Sum of Consecutive Terms of Pell and Related Sequences

Alexander J Zhu (lazostaf@gmail.com)
Alexander Gong (ag4668@columbia.edu)
Claremont Graduate University [Mentor:Stephanie Reyes]

Abstract of Report Talk: Linear Recurrences, which are discrete analogue of differential equations, arise in many areas of mathematics and science, and thus it is important to understand the properties of the terms of these sequences. Frequently there are fascinating relations among the elements, which reflect interesting behaviors; perhaps the most famous is the ratio of adjacent terms of the Fibonacci sequence ($F_n = F_{n-1} + F_{n-2}$) tends to the golden mean, which is ubiquitous throughout nature. Here we study new identities related to the sums of adjacent terms in the Pell sequence, which is defined by $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ and $P_0 = 0, P_1 = 1$ (though many of our results hold for more general recurrences).
We prove that the sum of N consecutive Pell numbers is an integer multiple of another Pell number if and only if $4 \mid N$ We also consider the Generalized Pell (k, i)-numbers defined by $p(n) = 2p(n-1) + p(n-k-1) \quad n \geq k+1$, with $p(0) = p(1) = \cdots = p(i) = 0$ and $p(i+1) = \cdots p(k) = 1$ for $0 \leq i \leq k-1$, and prove that the sum of $N = 2k+2$ consecutive terms is an integer multiple of another term in the sequence when $i = k-1$, and we prove that for the Generalized Pell $(k, k-1)$-numbers such an integer multiple relation does not exist when N and k are odd.

A model theoretic study of sparse graphs

Ava F Zinman (avazinman@college.harvard.edu)
Felix S Nusbaum (fsn3@williams.edu)
Towson University [Mentor:Miriam Parnes]

Abstract of Report Talk: **A model theoretic study of sparse graphs**

Felix Nusbaum
Zain Padamsee
Ava Zinman
Mentor: Dr. Miriam Parnes

A graph is "sparse" if its edge density is at most a certain upper bound. In particular, a graph G is called "α-sparse" if, for every subgraph of G, the ratio of vertices to edges is at least α. In this study, we consider graphs as structures in a language whose only element is a binary edge relation. This allows us to investigate model theoretic properties of the classes $\mathbb{K}_\alpha$, each of which consists of all α-sparse graphs.

We seek to characterize $\mathbb{K}_\alpha$ both directly and using the concept of boundary: a unique, minimal set made up of substructures of all graphs *not* in $\mathbb{K}_\alpha$. Other properties of interest include indivisibility, a coloring property related to the Ramsey property, and the amalgamation property, which addresses closure of classes under appropriate gluings.

We fully describe $\mathbb{K}_\alpha$ and construct its boundary for $\alpha \geq 1$. For $\alpha < 1$, we establish partial descriptions of the contents of and relationships between classes of α-sparse graphs. Furthermore, we find that $\mathbb{K}_\alpha$ is indivisible if and only if $\alpha > 2$, and has the amalgamation property if and only if $\alpha > \frac{3}{2}$.

Our model theoretic approach to sparse graph classes may allow us to generalize our results to many classes of structures in relational languages. In addition, graph sparsity has applications in computer science to efficient algorithms, data storage, and data access. We hope that our characterizations of sparse graphs may contribute to these generalizations and applications.