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Abstracts of Presentations

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Plenary Talks

In chronological order.

Topological invariance of non-topological invariants

Benson Farb
University of Chicago Friday, August 22nd @ 5:00 PM

In this talk I will explain three amazing theorems in geometry/topology, and how they relate to each other. These theorems helped form the current landscape in algebraic topology, differential topology, and differential geometry.

Patterns and randomness in dynamics and life

Amie Wilkinson
University of Chicago Saturday, August 23rd @ 9:30 AM

I will explore simple patterns encountered in a range of contexts and how they can be generated by dynamical systems.

Applications of exponential sums to solvability of equations, cryptography and coding

Ivelisse Rubio Canabal
University of Puerto Rico Saturday, August 23rd @ 5:10 PM

The p-divisibility of exponential sums of polynomials over finite fields can be used for a variety of applications. A little improvement on the estimation of the p-divisibility might be important in some applications. For example, the relation between the 2-divisibility of a Boolean function and certain deformations give information about cosets of Reed-Muller codes. Moreover, the computation of the exact p-divisibility of exponential sums associated to families of systems of polynomials guarantees the solvability of the system and, in the case of a Boolean functions,it proves that the functions are not balanced. In this talk we present the covering method, an elementary method to compute p-divisibility of exponential sums and see how it can be used in some applications.

Student Talks

In alphabetical order by the last name of the primary presenter.

On the Structure of Minimal $4(2k+1)$-orbits of Continuous Endomorphisms and Universality in Chaos

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Florida Institute of Technology	Mentor: Ugur Abdulla

We consider the problem on the structure of the periodic orbits of period $4(2k+1)$, $k = 1, 2, ...$ of the continuous endomorphisms on the real line which are minimal with respect to Sharkovski ordering. By developing the new method suggested recently in *Abdulla et al. J. of Diff. Equat. and Appl., 19,8(2013), 1395-1416,* it is proved that independent of k, there are 64 types of digraphs (and cyclic permutations) with accuracy up to inverse digraphs. The proof is constructive and we accomplish the construction of all 64 types of digraphs and cyclic permutations for minimal 12-orbits. We apply this result to the problem on the distribution of periodic windows within the chaotic regime of the bifurcation diagram of the one-parameter family of logistic type unimodal maps. First, we confirm through numerical analysis the conjecture made in a recent JDEA paper that the first two appearances of all the $2^n(2k+1)$-periodic windows with $k \geq 3$, as well as first appearances of $5 \cdot 2^n$- and $3 \cdot 2^n$-orbits while increasing the parameter are distributed according to the universal law

$$\cdots \to 2^n \cdot 11 \to 2^n \cdot 7 \to 2^n \cdot 9 \to 2^n \cdot 5 \to 2^n \cdot 7 \to 2^n \cdot 3 \to$$
$$\cdots \to 11 \to 7 \to 9 \to 5 \to 7 \to 3 \to \ldots \tag{1}$$

where the branches successfully follow from right to left as $n = 1, 2, ...$ Every orbit in (1) is universal, in the sense that it has a unique cyclic permutation and digraph independent of the unimodal map. In particular, the first appearance of $4(2k+1)$-orbit is always a minimal orbit, with precisely Type 1 digraph. The reason for the relevance of the Type 1 minimal orbit is the fact that the topological structure of the unimodal map with single maximum is equivalent to the structure of the Type 1 piecewise linear endomorphism. Yet another revelation of this research is the refinement of the universal law (1) for the third and fourth appearances of the periodic orbits. By employing the notation n_i for the ith appearance of the n-orbit, the following universal distribution of all the odd periodic orbits is revealed:

$$\ldots 17_2 \to 19_4 \to 13_1 \to 17_3 \to 15_2 \to 17_4 \to 11_1 \to 15_3 \to 13_2 \to 15_4 \to 9_1 \to$$
$$13_3 \to 11_2 \to 13_4 \to 7_1 \to 11_3 \to 9_2 \to 11_4 \to 5_1 \to 9_3 \to 7_2 \to 9_4 \to 3_1 \to \ldots$$

The same universal route is continued to the left for all four appearances of the $2^n(2k+1)$-orbits successfully for positive integers $n = 1, 2, ...$ Understanding the nature and characteristics of this fascinating universal route is an outstanding open problem for future investigations.

Unexpected distribution phenomenon resulting from Cantor Series Expansions

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University of North Texas Mentor: Bill Mance

We explore in depth the number theoretic and statistical properties of certain sets of numbers arising from their Cantor series expansions. As a direct consequence of our main theorem we deduce numerous new results as well as strengthen known ones.

This work has been submitted for publication. The pre-print is available at http://arxiv.org/abs/1403.3026.

Properties of some linear stability preservers

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Yale University Mentor: Anup Rao

We examine the movement of the roots of real-rooted polynomials, i.e. having all roots to be real numbers, $p(x) \in \mathbb{R}[x]$ after application of linear stability preservers $T : \mathbb{R}[x] \to \mathbb{R}[x]$. We focus on the $(1 - \frac{d}{dx})$ operator, which plays a role in the recent proof of the Kadison-Singer Problem by Marcus, Spielman and Srivastava, as well as the $T_! : x^k \to \frac{1}{k!}x^k, \forall k \in \mathbb{N}$ operator, providing a link between ordinary and exponential generating functions. Using majorization techniques we derive a bound on the size of the maximum root λ_{max} of the polynomial $T_!(p)$ obtained after applying $T_!$ on the polynomial $p(x)$. If $p(x) = \sum_{i=0}^{n} a_i x^i$ is a real-rooted polynomial, then we show that $\lambda_{max}(p(x)) \geq \frac{\lambda_{max}(T_!(p))}{\lambda_{max}(L_n(x))} \geq -\frac{a_{n-1}}{na_n}$, where $L_n(x)$ is the n-th Laguerre polynomial. The method used can be extended to give bounds on the movement of the maximum root of a big class of multiplier sequences. We also study the behavior of the spacings between the roots under applications of $(1 - \frac{d}{dx})^n$ to any real-rooted polynomial $p(x)$ and prove that, regardless of the initial distribution of the roots of $p(x)$, the gap between any two consecutive roots of $(1 - \frac{d}{dx})^n p(x)$ approaches infinity as n goes to infinity.

Euler Characteristics of p-Local Compact Groups

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Kansas State University Mentor: Alex Gonzalez

In 2010 Tom Leinster defined the Euler characteristic of a finite category in a manner compatible with the Euler characteristics of their classifying spaces. We present a generalization of this definition to the Euler characteristic of certain infinite categories, specifically those of linking systems of the p-local compact groups of Broto, Levi, and Oliver. This has applications to the fusion theory and p-local homotopy theory of compact Lie groups.

Spherical n-links

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This talk regards spherical links–that is, disjoint embeddings of 1-spheres and 0-spheres in S^2, where the notion of a split link is analogous to the usual concept. That is, a spherical n-link $\mathcal{L}$ is *split* if there exists an embedding ϕ of S^1 in $S^2 - \mathcal{L}$ such that each component of $S^2 - \phi(S^1)$ contains at least one piece (i.e. an embedded (0 or 1)-sphere) and each piece of $\mathcal{L}$ is entirely contained in one such component. Note that given this definition of a split link, we must first identify which pairs of points constitute 0-spheres. Our main result is a set of necessary and sufficient conditions for a spherical embedding of q circles and 2ℓ points to form a non split $(q + \ell)$-link given appropriate S^0 identifications: Given an embedding of q circles, the conditions for where we can place the 2ℓ points are as follows (where we consider fundamental groups before embedding the points):

1. Each simply connected region has at least one point.
2. A region R with fundamental group $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{\kappa-1}$ can have no more than $\ell - \kappa + 1$ points.
3. Given a region R as in (2), the other regions must have at least $2(\kappa - 1)$ points altogether.

We also give enumerations of distinct non split spherical n-links for $n \leq 8$.

The Chowla-Selberg Formula for Quartic Abelian CM Fields

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The Chowla-Selberg formula is a striking identity which relates values of the Dedekind eta function at quadratic points in the complex upper half-plane to products of values of Euler's gamma function at rational numbers. We will provide explicit analogues of the Chowla-Selberg formula for quartic abelian CM fields. This consists of two main parts. First, we implement an algorithm to compute the CM points at which we will evaluate a certain Hilbert modular function which generalizes the Dedekind eta function. Second, we exhibit families of quartic fields for which we can determine the precise form of the analogue of the product of gamma values. We will include several examples of our formulas for specific quartic fields.

Linking of n-spheres in simplicial complexes embedded in S^{2n+1}

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SUNY Potsdam/Clarkson Mentor: Joel Foisy

Define σ^n_M, to be the n-skeleton of a M-dimensional simplex. We generalize numerous results from spatial graph theory to higher dimensions, for $n > 1$. Conway, Gordon and Sachs showed that every spatial embedding of the complete graph on six vertices, K_6, contains a pair of cycles that form a non-split link and Taniyama along with others generalized this result to show that σ^n_{2n+3} must contain a non-split link of two n-spheres for every embedding into S^{2n+1} (note that $\sigma^1_5 = K_6$). In this talk we generalize these results to indicate what M is required to guarantee these non-split links to arbitrary non-split q-component links in every embedding of σ^n_M in S^{2n+1}, where we follow a similar procedure undertaken by C. Tuffley. In particular in an embedding, we prove that every n-face of σ^n_{3n+3} necessarily belongs to a sub-complex that consists of two non-split linked n-spheres. We also prove a stronger lemma than that of C. Tuffley's that allows for the construction of a new n-sphere once given two sets of non-split linked n-spheres that bridges the two, yielding a 3-component non-split link. Using this lemma we show that once we consider σ^n_{3n+3} joined at a n-face of another, we have a non-split three link and then further generalize this to an arbitrary collection of p simplices joined at a n-face of the next yielding a $(p+1)$-link within the resulting complex. Lastly, we prove a theorem that is stronger than C. Tuffley's in that we find a better lower bound than the one he provided for the requirement of the amount of vertices in our embeddings to yield arbitrary q-component non-split links. In the above language, this occurs for embeddings with the dimension of the M-simplex as $(2n+3)(q-1)$. From here we discuss how one might achieve a general construction for the uniform characterization of the minimum number of vertices required for any embedding of σ^n_M to contain a q-component non-split link.

The strange case of the Moebius power series: Reconciling theory and numerical evidence

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The Moebius function $\mu(n)$ is one of the most important functions in number theory whose behavior is closely related to the distribution of primes and the Riemann Hypothesis. We investigate the power series $f(z) = \sum_{n=1}^{\infty} \mu(n) z^n$, where $\mu(n)$ is the Moebius function and $|z| < 1$. The series was mentioned in Hardy and Littlewood's 1916 paper on the Riemann zeta function. A result of Delange implies that $f(r)$ is unbounded as $r \to 1-$. Yet, numerical data strongly suggest that $f(r)$ converges with limit -2. For example, at $r = 0.999$, 0.9999, 0.99999, 0.999999, $f(r)$ equals approximately -1.98810, -1.99880, -1.99988, -1.99998.

This apparent contradiction between the known theoretical behavior and numerical evidence was first pointed out in a little-known paper by Carl Froberg from the 1966. In this talk, we explain what lies behind this mystery, and we report on investigations on related phenomena. In particular, we consider variations of the Moebius power series, such as

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty} \mu(n) r^n, \qquad \sum_{\substack{n=1\\n\equiv a \bmod q}}^{\infty} \mu(n) r^n, \quad \sum_{n=1}^{\infty} \mu(n)\chi(n) r^n.$$

where a, q are positive integers and χ is a Dirichlet character. Using Mellin transform representations, we show that these series exhibit a similar type of "fake" asymptotics as the Moebius power series $f(r)$, and we obtain a heuristic formula that predicts the type of "fake" asymptotics for each of these series. These predictions agree closely with the observed behavior in numerical computations. This is joint work with Daniel Hirsbrunner and Dylan Yang.

Fibonacci sequence in knots and links

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Recent work by M. Cohen applies Dimer covers to the calculation of the Conway polynomial of knots. Given a knot (or connected link) diagram D, one can obtain the balanced overlaid Tait graph Γ, a bipartite planar graph whose vertex sets correspond to faces (with two omitted by placing stars in them) and edges in D. Dimer covers of Γ correspond to perfect matchings and are in bijective correspondence with clock states of D, as introduced by L. Kaufman. In particular the Dimer method provides a graphic method of producing the clock lattice of a knot diagram. In this presentation, we present a particular infinite sequence (L_n) of knots and links whose number of clock states is equal to the corresponding Fibonacci number. Notable knots and links among the (L_n) include the Hopf link, trefoil, figure eight knot and Whitehead link. Using the Dimer method, we find a recurrence relation, which mirrors somewhat the Fibonacci sequence, and allows for the Conway polynomial of any L_n to computed in terms of the previous elements.

The Igusa zeta function of a quadratic form over the p-adic integers

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Let $f(x_1, \ldots, x_n)$ be a quadratic form over the p-adic integers, $\mathbf{Z}_p$, and $N_i(f)$ be the number of zeroes of f modulo p^i. The Poincare series, $P(t) = \sum\limits_{i=0}^{\infty} \frac{N_i(f)}{p^{in}} t^i$, is a power series that organizes these zero counts. From the Poincare series one can obtain the Igusa local zeta function using the relation $Z(s) = p^s - (p^s - 1)P(p^{-s})$. For s in the complex right half-plane, the Igusa zeta function is defined to be the integral $Z(s) = \int\limits_{\mathbf{Z}_p^n} |f(x_1, x_2, \ldots, x_n)|_p^s dx_1 \ldots dx_n$, where $|\cdot|_p$ is the p-adic absolute value and $dx_1 \ldots dx_n$ is a volume element with respect to the Haar measure. We calculate both the Poincare series and the Igusa Zeta function for an arbitrary quadratic form over the p-adic integers where p is an odd prime. After determining the number of zeroes modulo p using generating functions, we use Hensel lifting to recursively calculate the Poincare series of f in terms of the Poincare series of simpler quadratic forms. Eventually this recursion stabilizes making the tail of the series geometric so that we may express $P(t)$ as a rational function. Our calculations correct the results of previous researchers.

Enumerating Cliques in Euler Totient Cayley Graphs

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For a positive integer n, let $G(\mathbb{Z}_n, \Phi)$ denote the Euler totient Cayley graph with n vertices. That is, $G(\mathbb{Z}_n, \Phi)$ is the graph with vertex set $\{0, 1, 2, \ldots, n-1\}$ and edge set $\{xy\colon x, y \in \mathbb{Z}_n, \gcd(x-y, n) = 1\}$. For each positive integer r, let S_r denote the r^{th} Schemmel totient function, a multiplicative arithmetic function defined by

$$S_r(p^\alpha) = \begin{cases} 0, & \text{if } p \le r; \\ p^{\alpha-1}(p-r), & \text{if } p > r \end{cases}$$

for any prime p and positive integer α. Maheswari and Madhavi have shown that if $n > 1$, then the number of triangles in $G(\mathbb{Z}_n, \Phi)$ is given by $\frac{1}{6}n\phi(n)S_2(n)$, where ϕ denotes Euler's totient function. We begin by establishing an interesting property of the Schemmel totient functions, and we using this property to generalize the formula given by Maheswari and Madhavi. We show that, for integers $m, n > 1$, the number of copies of K_m in $G(\mathbb{Z}_n, \Phi)$ is given by $\displaystyle\prod_{k=1}^{m} \frac{S_{k-1}(n)}{k}$, where we convene to let $S_0(n) = n$. As a corollary, we get the interesting divisibility relation $m! | \displaystyle\prod_{k=1}^{m} S_{k-1}(n)$.

Patterns and Combinatorial Statistics in Restricted Growth Functions

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We consider restricted growth functions (RGFs), which are words of length n in the positive integers. We define $w = a_1a_2\ldots a_n$ to be a restricted growth function if $a_1 = 1$ and for all $2 \le i \le n$, $a_i \le \max(a_1a_2\ldots a_{i-1}) + 1$. We let W_n be the set of restricted growth functions of length n. Restricted growth functions are an important set of words because they are in bijection with set partitions.

We examine pattern avoidance in restricted growth functions. We say that a word standardizes to another word if the relative ordering of the numbers is the same. We say that a word w *contains* another word v, called the pattern, if there is some subword w' of w that standardizes to v. If w does not contain v, we say that w *avoids* v. We let $W_n(v)$ be the set of words of length n that avoid the pattern v.

Wachs and White introduced four fundamental statistics on RGFs. For example, one such statistic is $lb(w) = \sum_i lb_i(w)$ where $lb_i(w)$ is the number of integers to the left of w_i which are also bigger than w_i. Letting q be a variable, we study the generating functions defined by $WLB_n(v) = \sum_{w \in W_n(w)} q^{lb(w)}$ where v has length 3 or 4, as well as the analogous polynomials for the other statistics. In particular, we determine their coefficients, degrees, and other properties. There are interesting connections with noncrossing set partitions, 2-colored Motzkin paths, integer partitions whose Ferrer's diagram fits in a box and other combinatorial objects.

Robustness of the Semicircle Law in Patterned and Sparse Matrix Ensembles

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Consider a Wigner ensemble of $N \times N$ real symmetric matrices; these matrices have their independent entries (i.e., the upper triangular components) drawn from a fixed probability distribution of mean 0, variance 1, and finite higher moments. Wigner's seminal result establishes that the limiting eigenvalue distribution for large matrices chosen randomly from this ensemble is a semicircle. Although new behavior can arise when additional structure is imposed on the ensemble, Wigner's semicircle law has been shown to be remarkably robust in various ways.

In our work, we prove that the semicircular law still holds when the Wigner ensemble is made sparse in several ways. Inspired by a general method of Bose and Sen for analyzing patterned matrix ensembles, we obtain our results by developing analogous techniques for sparse patterned matrix ensembles and using combinatorial and graph theoretic counting arguments. We prove that the semicircular law still holds when we set a large number $Z(N)$ of entries in the real-symmetric $N \times N$ matrices to be 0, or when the entries are set to 0 independently with probability $p(N)$, for $Z(N)$ and $p(N)$ satisfying modest growth conditions. We also analyze particular sparse ensembles with interesting arithmetic properties, such as ensembles of matrices whose eigenvalues are given by Kloosterman sums. We numerically explore their eigenvalue spacing distributions and extreme eigenvalue distributions, and we discuss progress toward elementary combinatorial proofs of the Sato-Tate law and the Weil bound for Kloosterman sums by applying the classical Moment Method and the theory of polynomial Legendre sums to these matrices.

A Quantum Gauss-Bonnet Theorem

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A generic curve in a surface is an immersed curve which has a finite number of transversal double points as its only singularities. There are several numerical homotopy invariants which capture important topological information about a generic curve, two of the most significant being the rotation number and the J^+ invariant. Recently, Lanzat and Polyak defined a polynomial (quantum) invariant of generic curves in the plane by integrating local geometric data, and showed that the rotation number and the J^+ invariant could be derived from their polynomial.

I extended their result to homologically trivial generic curves in closed oriented surfaces with Riemannian metric. Hopfs Umlaufsatz, which states that the integral of the curvature of a closed simple planar curve is $\pm 2\pi$, plays a crucial role in Lanzat and Polyaks paper, and their invariant is a quantum deformation of a formula for rotation number which can be seen as the Umlaufsatz with multiplicities. The Gauss-Bonnet Theorem plays an analogous role in my results. I give a new formula for the rotation number of a homologically trivial generic curve which generalizes the Gauss-Bonnet Theorem and can be seen as the Gauss-Bonnet Theorem with multiplicities. I then produce a quantum deformation of the previous invariant, analogous to Lanzat and Polyaks invariant for plane curves. I also show how the J^+ invariant can be derived from my invariant.

A New Convolution Theorem for Harmonic Mappings

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We will give a brief introduction to complex-valued, univalent (one-to-one) harmonic mappings, which are a generalization of complex analytic functions. There is interest in this field to combine harmonic mappings while preserving certain properties, which can be done with convolutions. Recently many papers have been published with results about convolutions of harmonic mappings. Our research involves obtaining conditions for which univalence is preserved through the convolution of harmonic mappings. Specifically, we will give necessary conditions for the convolution of two vertical strip mappings to be convex in the horizontal direction. We will discuss our new theorem with an outline of the proof along with examples. As well, current open questions related to our result will also be presented.

Zeros of Maass forms of Weight k

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Modular forms are complex analytic functions on the upper half plane satisfying certain symmetry conditions under Möbius transformations of $SL_2(\mathbb{Z})$. The zeros of modular forms have been well studied. For example, Rankin and Swinnerton-Dyer showed that all of the zeros in the fundamental domain $\mathcal{F}$ of the weight k Eisenstein series, E_k, lie on the bottom arc of $\mathcal{F}$. Maass forms are a certain generalization of modular forms, in which some of the conditions that modular forms are required to satisfy are slightly relaxed. We study the location of the zeros of the Maass form obtained by applying the Maass level raising operator $R_k = i\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) + \frac{k}{y}$ to E_k. We find that this Maass form has the same number of zeros on the bottom arc of $\mathcal{F}$ as E_{k+2}, and conjecture that all of its zeros in $\mathcal{F}$ lie on this arc. We note that this seems to hold for the Maass form obtained by applying the level raising operator multiple times to the Eisenstein series. Mathematica was used throughout our research mostly for experimentation, and sometimes to aid in symbolic calculations.

Holder estimates on Cauchy-type integrals in several variables

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Let U be a smoothly bounded domain in the complex plane with boundary Γ. The *Cauchy Integral* of a continuous function ϕ on Γ is the function $\mathcal{B}_1\phi$ given by:

$$\mathcal{B}_1\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\phi(t)dt}{t-z},$$

which is holomorphic on U and on $U^+ = \mathbb{C} \setminus \overline{U}$. The Cauchy Integral plays a central role in the study of boundary behavior of holomorphic functions. A fundamental result states that for each nonnegative integer k and each $0 < \alpha < 1$, the map $\mathcal{B}_1$ is continuous from $\mathcal{C}^{k,\alpha}(\Gamma)$ to $\mathcal{C}^{k,\alpha}(U)$ and also from$\mathcal{C}^{k,\alpha}(\Gamma)$ to $\mathcal{C}^{k,\alpha}(U^+)$, where $\mathcal{C}^{k,\alpha}$ denotes the Banach space of functions which are k times continuously differentiable and whose k-th partial derivatives are Hölder continuous with exponent α. The function ϕ can be recovered as the jump in the Cauchy Integral as one crosses the boundary Γ going from U to the unbounded domain U^+.

We consider the following n-dimensional analog of the Cauchy Integral:

$$\mathcal{B}_n\phi(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_\Gamma \frac{\phi(t)dt}{(t - z_1)(t - z_2) \ldots (t - z_n)},$$

where the smooth curve Γ is the boundary of a domain U in the plane, ϕ is continuous on Γ and $\mathcal{B}_n\phi$ is a function of n complex variables. Then $\mathcal{B}_n\phi$ is a symmetric holomorphic function on U^n, and in fact on $(\mathbb{C} \setminus \Gamma)^n$. The map $\mathcal{B}_n$ arises in the study of boundary regularity of some proper holomorphic maps in several variables. We prove the following result:

Theorem 1. *For $k \geq 0$ and $0 < \alpha < 1$, the map $\mathcal{B}_n$ is continuous from $\mathcal{C}^{k+n-1,\alpha}(\Gamma)$ to $\mathcal{C}^{k,\alpha}(U^n)$.*

It may be a little surprising that $\mathcal{B}_n$ defined by a kernel analytic in z should display loss of smoothness at the boundary. Our proof uses a representation of the map $\mathcal{B}_n$ in terms of discrete analogs of the derivative known as *divided differences*.

We also characterize the range of the map $\mathcal{B}_n$, study the jump behavior of $\mathcal{B}_n\phi$ as one crosses the boundary of U^n, and show that in contrast to the case $n = 1$, the Cauchy principal value of $\mathcal{B}_n\phi$ does not in general exist at a boundary point of U^n for $n \geq 2$ and $\phi \in \mathcal{C}^{0,\alpha}(\Gamma)$.

Knot Depth for Positive Braids

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Given a projection of a knot, one can alter it at a crossing using what is called the *oriented skein relation*. This relation takes this projection, and considers instead the two new knot projections which result by (a) changing which strand in the crossing is on top, and (b) removing the crossing in a way consistent with the orientation. The depth of a knot is essentially the minimal number of levels in a resolution tree for that knot using the oriented skein relation. The depth of a knot is related to the complexity of the knot's Alexander polynomial and, more generally, any polynomial invariant which satisfies the skein relation. Every knot can be represented by a word in a group called the braid group. We study the depth of knots representable by positive braids—a class including all torus knots and links. We have proven that the depth of a knot represented by a positive braid is $l - n + 1$, where l is the length of the braid word and n is the number of strands in the braid.

A Measure of Closeness using the Distance between Two Finite Groups

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Our research explores the notion of distance between finite groups, a measure of how close these groups are to being isomorphic. This concept relies heavily on what we call matching pairs, since they give us a measure of closeness between the two finite groups. In order to find a precise formula for the distance and understand its properties, we looked at the metric properties of the distance as well as the validity of some of Mark Lewers conjectures. Overall, we proved Lewers hypothesized upper bound for the distance between $\mathbb{Z}_m$ and $\mathbb{Z}_n$, discovered a necessary condition to obtain the distance, and gave explicit formulas for computing the distance between finite cyclic groups of small order. Our results suggest that this distance has all the properties of a metric except the triangle inequality. We have also discovered multiple lower bounds for the distance between $\mathbb{Z}_m$ and $\mathbb{Z}_n$. This has motivated us to look for even stronger upper and lower bounds for the distance in order to better approximate the true value of the distance, prove the triangle inequality, and discover which finite groups have greater structural similarity.

Pointwise Quasisymmetric Minimality and Uniform Cantor Sets

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We say that a set $E \subset \mathbb{R}$ is (quasisymmetrically) minimal if $\dim_H f(E) \geq \dim_H E$ for every quasisymmetric mapping f of the real line. By a theorem of Kovalev it is known that minimal subsets of the line have to be of Hausdorff dimension 1. In dimension 1 there are full measure sets which are not minimal (first constructed by Tukia in 1989). On the other extreme, there are zero measure sets which are minimal (first constructed by Hakobyan in 2006).

In this talk we will give sufficient conditions for subsets of the line to be quasisymmetricaly minimal. The main novelty of our approach is a notion of pointwise minimality for measures. We use this notion to show that *uniform Cantor sets are minimal for quasisymmetric mappings of the line if and only if they have Hausdorff dimension 1.* This answer a question of Hu and Wen from 2008.

On a Variant of the Lang-Trotter Conjecture Involving Binomial Elliptic Curve Coefficients

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Let E be an elliptic curve $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Z}$. We can associate to E an L-function $\sum_n a_E(n)/n^s$, where for almost all n equal to a prime p we have $a_E(p) = p - \#E_p$, with $\#E_p$ the number of solutions $(x, y) \in \mathbb{Z}/p\mathbb{Z}$ to $y^2 \equiv x^3 + ax + b \bmod p$. This L-function encodes many important properties of the elliptic curve; the famous Birch and Swinnerton-Dyer conjecture states that its order of vanishing at $s = 1/2$ equals the rank of the group of rational solutions. On average, we expect $\#E \approx p$ with fluctuations about this expected value given by the trace of the Frobenius operator. It is known that the deviation is bounded in absolute value by $2\sqrt{p}$, and thus $a_E(p) \in (-2\sqrt{p}, 2\sqrt{p})$.

In 1976, Lang and Trotter conjectured an asymptotic formula for $\pi_{E,r}(x)$, the number of primes p up to x for which $a_E(p) = r$ for any fixed r: $\pi_{E,r}(x) \sim C_{E,r} \frac{\sqrt{x}}{\log x}$ as $x \to \infty$, for some constant $C_{E,r}$. While this question is well beyond current methods, in 2006, James and Yu developed an asymptotic formula for the density of traces of Frobenius that are kth powers. Their analysis uses the Hardy-Littlewood Circle Method, in particular, estimates for sums of generating functions of pure powers. A natural extension of their work is to examine polynomials that are not pure k-th powers; thus instead of trying to solve $a_E(p) = n^k$ we want $a_E(p) = f(n)$ for some polynomial f. We discuss our result on extensions to large classes of functions. For example, we prove how often $a_E(p)$ lies in a given arithmetic progression, or is a triangular number (i.e., a binomial coefficient $\binom{n}{2}$). Doing so requires a delicate analysis of new generating functions on the minor arcs in order to obtain sufficient cancellation; these results are of independent interest for other problems in number theory.

A Family of Rank 6 Elliptic Curves over Number Fields

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The points on an elliptic curve $\mathcal{E}$ over a number field K carry the structure of a finitely generated abelian group of the form $\mathcal{E}(K) = \mathbb{Z}^r \oplus T$. The number r is the *rank* of $\mathcal{E}$ and T is a finite group. Studying the rank of elliptic curves is important in modern number theory, with applications from cryptography to computing class numbers.

We construct a family of elliptic curves over a number field K, and prove that when K is Galois over $\mathbb{Q}$, each curve has rank six. Unlike most constructions, which only bound the rank, we find the rank exactly. By evaluating Legendre sums, we determine equations for curves $\mathcal{E}$ with $a_{\mathfrak{p}}(\mathcal{E}) = 6$, where $a_{\mathfrak{p}}(\mathcal{E})$ measures the number of solutions over $\mathcal{O}_K/\mathfrak{p}$ for a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$. The rank is equal to a sum over all $\mathfrak{p}$ by a theorem of Rosen and Silverman. We control the asymptotic behavior of this sum with Chebotarev's density theorem, and demonstrate that it converges to $a_{\mathfrak{p}}(\mathcal{E})$. In fact, we obtain in this manner not only infinitely many elliptic curves over K, but also infinitely many elliptic surfaces, i.e., elliptic curves over the function field $K(T)$.

Additionally, we hypothesize that curves defined analogously over non-Galois extensions $L/\mathbb{Q}$ also have rank six, which we prove in several cases, and determine bounds for all other cases. Moreover, we prove that when $K = \mathbb{Q}$, if there are any points of finite order in $\mathcal{E}(\mathbb{Q})$, they must have order three. However, we are able to modify our construction to find a family of curves with group $\mathcal{E}(\mathbb{Q}) = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$. This generalizes work of Arms, Lozano-Robledo, and Miller, which only dealt with families over $\mathbb{Q}$.

Product Laurent Phenomenon Algebras

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Cluster algebras are a rather new algebraic object, introduced by Fomin and Zelevinsky in 2001 and notable for their interesting combinatorial structure. These objects were generalized to Laurent Phenomenon (LP) algebras in 2012. In these structures, one usually starts with some initial seeds, a collection of variables $\mathbf{x} = \{x_1, \ldots, x_n\}$ and associated exchange polynomials $\mathbf{F} = \{F_1, \ldots, F_n\}$. One can then generate new seeds via a process called mutation. We deduce the structure of the algebra generated from the initial seed given by exchange polynomials $F_i = A_i + \prod_{j \neq i} x_j$ over $\mathbb{Z}[A_1, \ldots, A_n]$ and compare it to the algebra generated by initial seed $F_i = A_i + \sum_{j \neq i} x_j$. Our algebra displays interesting recursive structures that give insight into similar LP algebras and bases of their underlying commutative ring.

The Complex Zeros of a Gaussian Random Polynomial

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Let $P_n(z) = \eta_0 + \eta_1 z + \eta_2 z^2 + \ldots + \eta_{n-1} z^{n-1}$ be a Gaussian random polynomial of degree $n-1$, where $\eta_0, \ldots, \eta_{n-1}$ are independent normalized Gaussian random variables and $z \in \mathbb{C}$. Let further Ω be any Lebesgue measurable subset of $\mathbb{R}$ and denote by $\nu_n(\Omega)$ the number of zeros in Ω of $P_n(z)$. In 1943, Kac obtained an explicit intensity function $g_n(x)$ for which the ensemble average of $\nu_n(\Omega)$ is given explicitly by $\int_\Omega g_n(x)\,dx$ for each positive integer n. In 1995, Shepp and Vanderbei extended Kac's result to the case when Ω is any Lebesgue measurable subset of $\mathbb{C}$. In this paper, we consider the case when the coefficients $\eta_0, \ldots, \eta_{n-1}$ of $P_n(z)$ are independent complex normalized Gaussian random variables and employ the method of Shepp and Vanderbei to obtain a simplified explicit intensity function $h_n(z)$ for which the ensemble average of $\nu_n(\Omega)$ is given explicitly by $\int_\Omega h_n(z)\,dz$ for each positive integer n. We also provide numerical computations that exhibit the behavior of $h_n(z)$ for various values of n and obtain the limit of $h_n(z)$ by letting n tends to infinity.

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Ramsey Theory Problems in Z: Avoiding Generalized Progressions

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Ramsey theory is frequently concerned with seeing how large of a set one can take without a certain structure being forced to be present somewhere. Two well-studied Ramsey-theoretic problems are to consider subsets of the natural numbers with no three elements in arithmetic progression, or geometric progression. We study new generalizations of this problem, where we vary the kinds of progressions that are avoided and the metrics used to evaluate the density of the resulting subsets.

We can view a three-term arithmetic progression as a sequence $x, f(x), f(f(x))$, where $f(x) = x + k$ with k a nonzero integer. Thus a set avoiding a three-term arithmetic progression is equivalent to never having three elements of the form $x, f(x), f(f(x))$ with $f \in \mathcal{F}_{\mathrm{t}}$, the set of integer translations. Similarly, we can construct other sequences in terms of different families of functions. We investigate several families, including geometric progressions ($f(x) = nx$ with $n > 1$ a natural number) and exponential progressions ($f(x) = x^n$). Progression-free sets are often constructed "greedily" by beginning with 1 and taking every number so long as it is not in progression with any of the previous elements. Rankin characterized the greedy geometric set in terms of the greedy arithmetic set. We characterize the greedy exponential set and prove that it has asymptotic density 1, and then discuss how the optimality of the greedy set depends on the family of functions used to define progressions.

Traditionally, progression-free sets are judged by their asymptotic density. We consider what happens when we change our metric to other densities. We analyze sets with high analytic, multiplicative, and uniform density and how the type of density changes the structure of optimal sets.

Universal Lower-Order Biases in Elliptic Curve Fourier Coefficients

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An elliptic curve $E_{a,b}$ over the rationals are all pairs (x, y) that solve $y^2 = x^3 + ax + b$ for a fixed pair $a, b \in \mathbb{Z}$. These well-studied objects appear in many places in mathematics, from the theory of Diophantine equations to cryptography. The subset of solutions with rational coordinates form a finitely generated abelian group, whose rank is conjecturally related to the number of solutions modulo p; we write the number of solutions modulo p as $p - a_E(p)$. We can use these coefficients to build an L-function by setting $L(s, E_{a,b}) = \sum_n a_{E_{a,b}}(n)/n^s$, and many properties of the elliptic curve are encoded in this function. Instead of one elliptic curve, we can study a one-parameter family of curves over $\mathbb{Q}(T) : \mathcal{E} : y^2 = x^3 + A(T)x + B(T)$, where now $A(T), B(T)$ are polynomials in $\mathbb{Z}[T]$ and each specialization of T to an integer t gives an elliptic curve over $\mathbb{Q}$. Let $A_{r,\mathcal{E}}(p) := \sum_{t \bmod p} a_{E_t}(p)^r$ be the r^{th} moment of the Fourier coefficients of the associated L-functions. As the first moments are related to the rank of the family over $\mathbb{Q}(T)$, one of the most important quantities associated to an elliptic curve, it is natural to explore the distribution and consequences of the second moment.

Michel proved that $A_{2,\mathcal{E}}(p) = p^2 + O(p^{3/2})$ for families without complex multiplication, and cohomological arguments prove that the lower order terms are of sizes $p^{3/2}, p, p^{1/2}$ and 1. We have extensively studied thousands of families numerically and theoretically, and in each case the first term in the second moment expansion that does not average to zero has always had a negative bias. We conjecture that this bias always exists, and using methods from algebraic geometry and the theory of Legendre sums, we are able to prove this claim for many families. In particular, we consider families with rank and families with unusual distributions of signs. These non-trivial cases strongly support our bias conjecture.

The observed and proven negative bias of the lower order terms has implications towards the excess rank conjecture and the behavior of the zeros near the central point of elliptic curve L-functions. In 1998 Rosen and Silverman proved a conjecture of Nagao that the first moment $A_{1,\mathcal{E}}(p)$ is related to the rank; we end by formulating an analogous conjecture for the second moment, which we prove in some cases.

P-colorability by pairs as a knot invariant

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3-colorability is a well-known knot invariant that is very easy to construct while also being very limited in its use. In particular, it does not distinguish a knot from its mirror image. Recall that a knot diagram is tri-colorable if you can color all of its strands using at least 2 or at most 3 colors such that at each crossing, either the 3 different colors come together or the crossing only has 1 color. In 2011, Roger Fenn defined the 3-colorability by pairs as a knot invariant that allowed him to show that the right trefoil is different from the left trefoil.

In this research project, we first used 3-colorability by pairs to show that the 6-1, 7-7, and 8-5 knots are chiral (namely different from their mirror image). This invariant's use is limited to knots that are 3-colorable in the classic meaning. We therefore proceed to define p-colorability by pairs (such that p is any prime number) and proved that it is a knot invariant (namely it is unchanged by all 3 Reidemeister moves). The invariant for p-colorabilty by pairs (as for 3-colorabilty defined by Fenn) is not just the coloring. The useful invariant is the list of crossings that appear in the knot diagram. The list of all possible crossings through the p-coloring gives us the structure of an abelian group, which allows us to show that it is a knot invariant. As an example, we show that the 5-1 and 7-4 knots, both 5-colorable, are also chiral.

Note that all of the knots mentioned above were already known to be chiral but it required much more involved topological tools such as the Kauffman Polynomial among others.

Currently, we are studying the effect of the composition of knots on p-coloring by pairs and we conjecture that this technique can not be used for composite knots.

A generalized Newman's conjecture for function field L-functions

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De Bruijn and Newman introduced a deformation of the completed Riemann zeta function $\zeta(s)$, and proved there is a real constant Λ which encodes the movement of the nontrivial zeros of $\zeta(s)$ under the deformation. The Riemann hypothesis (RH) is equivalent to the assertion that $\Lambda \leq 0$. Newman, however, conjectured that $\Lambda \geq 0$, remarking, "the new conjecture is a quantitative version of the dictum that the Riemann hypothesis, if true, is only barely so."

Andrade, Chang, and Miller extended the machinery developed by Newman and Polya to L-functions for function fields, which are the analogue of ζ for fields $\mathbb{F}_q(T)$. In this setting we must consider a modified Newman's conjecture: $\sup_{f\in\mathcal{F}} \Lambda_f \geq 0$, for $\mathcal{F}$ a family of L-functions. We extend their results by proving this modified Newman's conjecture for several families of L-functions. In contrast with previous work, we are able to exhibit specific L-functions for which $\Lambda = 0$, and thereby prove a stronger statement: $\max_{L\in\mathcal{F}} \Lambda_L = 0$. Using analytic techniques, we show a certain deformed L-function must have a double root, which implies $\Lambda = 0$. For a different family, we construct particular elliptic curves with $p+1$ points over $\mathbb{F}_p$. By the Weil conjectures (which have been proven), this has either the maximum or minimum possible number of points over $\mathbb{F}_{p^{2n}}$. The fact that $\#E(\mathbb{F}_{p^{2n}})$ attains the bound tells us that the associated L-function satisfies $\Lambda = 0$.

On Expandability of Discrete Analytic Functions

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Discrete function theory is a branch of mathematics which deals with functions defined on a discrete set of points, such as the vertex set of a graph. The class of discrete analytic function is the discrete counterpart of the classical complex analytic functions. A noticeable peculiarity of discrete complex analysis is that the point-wise product of discrete analytic functions is not discrete analytic in general; for example, on the integer lattice in the complex plane, the functions z and z^2 are discrete analytic, but the function z^3 is not. Thus it is not an easy task to describe even the simplest algebraic discrete analytic functions, such as polynomials, rational functions or, even less, power series. Over the years, there have been developed bases of discrete analytic polynomials to approximate discrete analytic functions (the analogue of Weierstrass approximation), and formal power series associated to discrete analytic functions, but only very recently was there success developing a polynomial basis that allows power series expansion of discrete analytic functions (which would be the discrete analogue of Taylors expansion). In this project, we formalize the construction of this basis using a more natural definition of expandability. We then use this expansion to characterize discrete analytic functions that are rational with respect to a suitable, non-point-wise, product of discrete analytic functions which preserves analyticity.

Analysis of Interfaces for the Nonlinear Diffusion Equation with Linear Convection

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We investigate the problem of interface development in a Cauchy problem for the nonlinear diffusion-convection equation

$$u_t = (u^m)_{xx} + bu_x, x \in \mathbb{R}, t > 0; \quad u(x,0) = C(-x)_+^{\alpha}, x \in \mathbb{R},$$

where $m, \alpha, C > 0, b \in \mathbb{R}$. This problem arises in physics, biology, and chemistry; examples of applications include heat radiation in plasma, the spatial spread of biological populations, and the diffusion of chemicals through groundwater. The physics of the situation indicates that the direction of the movement of the interface is an outcome of the competition between nonlinear diffusion vs. linear convection. The problem of determining the short-time behavior for interfaces of nonlinear diffusion equations, known as a Barenblatt problem, was first formulated in the 1950s. A full solution of the Barenblatt problem for the reaction-diffusion equation was given in 2000 [Abdulla and King, SIAM J. Math. Anal., 32, 2(2000), 235-260] and 2002 [Abdulla, Nonlinear Analysis, 50, 4(2002), 541-560], but the problem remains open for the reaction-diffusion-convection equation.

It is proved that for the opposing direction of convection ($b > 0$) depending on m, α and C, the interface may initially expand or shrink. For the slow diffusion case ($m > 1$), the borderline case in the parameter space (m, α) is given through the curve $\alpha = 1/(m-1)$. The interface expands if $\alpha < 1/(m-1)$ and shrinks if $\alpha > 1/(m-1)$. The behavior of the interface in the borderline case depends on the constant C. There is a critical value C_* such that the interface expands if $C > C_*$ and shrinks if $C < C_*$. In the latter case, the explicit global traveling wave solution is found. We identify the region in the parameter space where a global self-similar solution exists, and in particular, the direction of the interface changes in time: a so called turning interface phenomenon is observed. For the cooperating direction of the convection ($b < 0$), the interface always expands and an explicit formula for the interface and local solution is derived in the whole parameter space. For the fast diffusion case $m < 1$, there is an infinite speed of propagation. In this case, we derive that the asymptotics of the solution at infinity agree with those of the diffusion equation. A WENO numerical scheme was applied to the problem and numerical results support our proved estimations.

Cops and Robbers on Generalized Petersen Graphs and I-Graphs

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The game of Cops and Robbers is a perfect information, vertex-pursuit game in which a set of "cops" and a "robber" occupy vertices in a graph G. The cops place on vertices of the graph, followed by the robber and then both alternate turns, either moving to adjacent vertices or passing. We say a graph is k-cop win if k cops playing on a graph ensures that, in a finite number of moves, one of the cops can move onto the same vertex as the robber. The cop number of a graph is the least such k.

We study the game of Cops and Robbers played on the infinite family of graphs known as Generalized Petersen graphs, defined as follows. Let n and k be positive integers such that $n \geq 5$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. The Generalized Petersen graph $GP(n, k)$ is the undirected graph with vertex set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ and the following edges: $(a_i, a_{i+1}), (a_i, b_i)$, and (b_i, b_{i+k}) for each $i = 1, \ldots, n$ with indices read modulo n.

To bound the cop number of $GP(n, k)$, we lift the game to an infinite cyclic cover, $GP(\infty, k)$, where the latter has vertex set $\{a_i \mid i \in \mathbb{Z}\} \cup \{b_i \mid i \in \mathbb{Z}\}$ and edges defined analogously. There is a graph homomorphism $\pi : GP(\infty, k) \to GP(n, k)$ defined by reducing indices modulo n. Playing a variant of the game on $GP(\infty, k)$, we show that the cop number of every Generalized Petersen graph is less than or equal to 4. With the aid of a computer, we have verified that there exist Generalized Petersen graphs realizing this bound.

Additionally, we generalize our results to all I-graphs. These graphs are similar to Generalized Petersen graphs, except that two parameters j and k define the adjacencies: (a_i, a_{i+j}), (a_i, b_i), and (b_i, b_{i+k}). A lifting strategy is used to prove that the cop number of any I-graph is less than or equal to 5.

Bounds for self-intersection numbers of closed curves on the 2-punctured plane

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The homotopy classes of closed curves on a topological space form a group called the fundamental group. The fundamental group of the 2-punctured plane is free on two generators where each generator is a simple closed loop around a puncture; we will denote these generators by a and b. In this project, we address the problem of finding the minimum number of self-intersections of closed curves of a free homotopy class in the 2-punctured plane.

In 2010, Chas, M., & Phillips, A. provided a bound for the minimum possible number of self-intersections of any element in the group. Although this bound is powerful because of its generality, often it is also very wide.

In this research project, we constructed algorithms that establish tighter upper bounds for two types of elements of the fundamental group:

1. For elements composed of only the two generators (and not their inverses), all of which take the form $a^{a_1} b^{b_1} \cdots a^{a_n} b^{b_n}$, the self intersection number is bounded by $\sum_{i=1}^{n} (2i - 1)(a_i + b_i - 2) + n - 1$.

2. For elements of the form $(a^s b^{-t})^n$ the self intersection number is bounded by $\frac{(s+t)(n^2+n)}{2} - n^2 + n - 1$

where n, s, t, and all a_i and b_i are positive integers.

Additionally, we conjecture that in certain special cases these algorithms yield the exact minimum number of self-intersections (which is still an open problem).

An inequality between the collective and individual times for completing a job

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The inequality between the arithmetic and harmonic means of n positive numbers, can be interpreted as the fact that the time necessary for n workers, each working at a constant speed (productivity), to complete a job working together, is less than or equal to $1/n^2$ of the sum of the individual times required by the workers laboring alone to complete that job. We show first that if we consider n workers, who are becoming tired in time, and whose productivity decreases exponentially in time at a rate that depends only on the difficulty of the job performed, but not on the workers, then the above inequality still holds. The proof relies on Jensen inequality. Finally, we extend this result to the case in which we have n workers, becoming tired continuously in time, and whose order of productivity remains the same in time, that means, if worker 1 productivity is greater than or equal to worker 2 productivity at time 0, then worker 2 productivity never exceeds worker 1 productivity.

Explicit bounds for the ϵ-pseudospectra for certain classes of operators

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The concept of ϵ-pseudospectrum generalizes the concept of the spectrum of an operator: for an operator A, we define the ϵ-pseudospectrum to be the set of points $z \in \mathbb{C}$ where $\|(zI - A)^{-1}\| > 1/\epsilon$. This set, which necessarily contains the spectrum, can be used to measure how stable the spectrum of an operator is under small perturbations of the operator, a question with central importance in numerical analysis. It is widely known that normal operators have particularly well-behaved pseudospectra, while non-normal operators exhibit more pathological behavior and show large growth of the resolvent norm $\|(zI - A)^{-1}\|$ for z far away from the spectrum of A.

While asymptotic bounds are known for the norm of the resolvent for many classes of non-normal matrices, explicit formulae for the ϵ-pseudospectra remain largely absent from the literature. Our work addresses this gap by explicitly describing the algebraic curves that bound the ϵ-pseudospectra of 2×2 matrices with complex entries. Through this characterization, we observe two different classes of behavior for diagonalizable and non-diagonalizable matrices; we also observe that for matrices with distinct eigenvalues, the ϵ-pseudospectrum grows as $|\lambda_1 - \lambda_2| \cot \theta$ for small θ, where θ is the angle between the vectors forming the columns of the matrix. We also find formulae characterizing the ϵ-pseudospectra of $n \times n$ periodic bidiagonal matrices for general n. Finally, we use these general finite-dimensional techniques to understand the asymptotic behavior of the radii of the pseudospectra of compact operators on Hilbert spaces.

Complex Ramsey Theory

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Optimal bounds in Ramsey theory usually require the construction of the largest sets not possessing a given property. One problem which has been investigated by many concerns the construction of the largest possible subset of $\{1, 2, \ldots, n\}$ (as $n \to \infty$) that are free of 3-term geometric progressions. Building on the significant recent progress in constructing such large sets, we consider higher dimensional analogues.

Specifically, let $\mathbb{Z}[i] = \{a + ib, a, b \in \mathbb{Z}\}$ (with $i = \sqrt{-1}$) be the Gaussian integers. We consider analogous questions in this setting. The construction is more involved now as there are significantly more ways to form a progression. We derive sets of Gaussian integers that avoid 3-term geometric progressions by utilizing the relationship between norms and powers of primes in the norms' factorization. We analyze how our choice of allowed ratios affects the problem – pure integer ratios lead to a projection of the original problem into two dimensions, whereas Gaussian integer ratios create rotation and dilation in the complex plane. Motivated by Rankin's canonical greedy set over the integers, we construct a set of Gaussian integers avoiding Gaussian integer ratios in a similar fashion, and compute its density using Euler products and the Riemann zeta function. This density turns out to be significantly lower than the density in the case of the integers. We also establish upper and lower bounds on the maximum upper density of such sets. Finally, we discuss our extensions to other quadratic number fields, and the dependence of the density on the structure of the number field (in particular, on the norm and class number).

Large gaps between zeros of $GL(2)$ L-functions

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The distribution of critical zeros of the Riemann zeta function $\zeta(s)$ and other L-functions lies at the heart some of the most central problems in number theory (under the Generalized Riemann Hypothesis these zeros are of the form $1/2 + i\gamma$ with γ real). The Euler product of $\zeta(s)$ translates information about its zeros into knowledge about the distribution of the prime numbers; similar arithmetically important results (such as the ranks of elliptic curves and the size of the class number) is encoded for other L-functions. A natural question to ask is how often large gaps occur between critical zeros of L-functions relative to the normalized average gap size. A striking connection to random matrix theory suggests that the spacing distributions between zeros of many classes of L-functions behave statistically similarly to the spacing distributions of eigenvalues of large Hermitian matrices. In particular, it is believed that gaps as large as arbitrarily many times the average gap between zeros occur infinitely often. However, few nontrivial results in this direction have been established.

Through the work of many researchers, the best result to date for $\zeta(s)$ is that gaps at least 2.9 times the mean spacing occur infinitely often, assuming the Riemann hypothesis. In the present work, we prove the first nontrivial result on the occurrence of large gaps between critical zeros of L-functions associated to primitive holomorphic cusp forms f of level one. Combining mean value estimates from Montgomery and Vaughan and extending a method of Ramachandra, we develop a procedure to compute shifted second moments, which are of interest to other questions besides our own. Using the mixed second moments of derivatives of $L(1/2+it, f)$, we prove that there are infinitely many gaps between consecutive zeros of $L(s, f)$ on the critical line which are at least $\sqrt{3}$ times the average spacing. Our techniques are general and promise similar results for other primitive $GL(2)$ L-functions such as L-functions associated to Maass forms.

Rotation Number of Outer Billiard with Polygonal Invariant Curves

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Outer billiard is a simple dynamical system introduced by B. Neumann and popularized by J. Moser in the 1970's. This paper studies rotation number on the invariant curve of a one parameter family of outer billiard tables. Given a convex polygon η, we can construct an outer billiard table $\mathcal{T}$ by cutting out a fixed area $\mathcal{A}$ from the interior of η. $\mathcal{T}$ is piecewise hyperbolic and the polygon η is an invariant curve of $\mathcal{T}$ under the billiard map ϕ. We prove that, if $\beta \in \eta$ is a periodic point under ϕ with rational rotation number $\tau = \frac{p}{q}$, then ϕ^q is not the local identity at β. This shows that the rotation number τ as a function of the parameter $\mathcal{A}$ is a devil's staircase function, which implies that there exists periodic orbits of arbitrary period on the polygonal invariant curve. However, this phenomenon is not universal in billiard systems. For example, N. Innami [1989] constructed a family of smooth inner billiards with invariant curves that consist only of 3-periodic orbits, while Y. Baryshnikov/V. Zharnitsky [2006] showed that there exist billiard tables which consist only of periodic points but have no elliptic boundaries.

A Theory on Abelian Difference Sets

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Difference sets in modern algebra have many applications in fields as diverse as signal design, cryptography, and DNA analysis. We investigate the theoretical proof for the existence of certain difference sets.

Let $(G, +)$ be an abelian group of order v and let D be a subset of G containing k elements. Consider the list of differences $(x - y(\text{mod} v)|x, y \in D, x \neq y)$. If every nonzero element of G appears exactly λ times, then D is known as a (v, k, λ) difference set in G.

An H-developed matrix satisfies the condition that $a_{gh} = a_{g+k,h+k}$ for all g, h, k in H, and a weighing matrix $W(v, k)$ is a square matrix of size v all of whose entries lie in $\{0, 1, -1\}$ satisfying $WW^{-1} = k_v I_v$, where I is the $v \times v$ identity matrix.

Using a result of Arasu and Ma (1998), we prove the following theorem: Let $G = \langle \alpha \rangle \times H$ be an abelian group of order $v = p^t w$ where p is an odd prime, $(p(p-1), w) = 1$, $o(\alpha) = p^t$, and $|H| = w$. If G contains a (v, k, λ) difference set where $k - \lambda = p^{2r}$, then there exists an H-developed weighing matrix of weight p^{2r}.

Previously published work proving the non-existence of the $(429, 108, 27)$ difference set in Z_{429} was machine-dependent; here, we provide the first theoretical proof of such an object. In addition, we generalize the previous theorem that holds for odd prime powers and extend it to account for both even and odd powers of r.

Poster Presentations

In alphabetical order by the last name of the primary presenter.

One-level density for cusp forms of prime square level

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Recent work strongly supports the Katz-Sarnak philosophy that, in the correct asymptotic limit, matrices from the classical compact matrix groups accurately model the statistics of zeros of L-functions. In particular, by studying distributional statistics of critical zeros lying close to the central point, we can distinguish the symmetry group attached to a given family of L-functions (under the Generalized Riemann Hypothesis the critical zeros are of the form $1/2 + i\gamma$ with γ real). These low-lying zero statistics are known as the n-level densities. In 2000, Iwaniec, Luo and Sarnak (ILS) published the first major paper to calculate the one-level density for families of L-functions. They proved that, for suitable test functions, the one-level density for families of primitive holomorphic cuspidal newforms of weight k and squarefree level N agrees with the particular orthogonal symmetry group conjectured for the family as kN tends to infinity.

The restriction in ILS that N be squarefree greatly simplifies the argument, though it is widely expected that the main term (and therefore the limiting behavior) of the one-level density for cuspidal newforms is indifferent to the squarefreeness of the level, though lower order terms can differ due to the different arithmetic of the family. We derive the one-level density for prime square level, prove that it converges to the correct scaling limit, and analyze the lower order correction. Our analyses requires several new techniques, especially to determine and work with the bases for these spaces and to derive tractable versions of the Petersson formula to average over the families.

The Emergence of 4-Cycles Over Extended Integers

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Given a ring R and a polynomial f in $R[x]$, an n-cycle is a sequence of n elements of the ring, $(x_1, \ldots, x_n)$, such that $f(x_1) = x_2, f(x_2) = x_3, \ldots, f(x_n) = x_1$. If we consider polynomials in $\mathbb{Z}[x]$, we can quickly see that long cycles are hard to find. In fact, it turns out that over the integers, the only possible cycle lengths are 1 and 2. However, adjoining elements of the form $1/p$ with p prime to our ring of integers is known to sometimes introduce 4-cycles. To determine whether adjoining certain sets of prime reciprocals will introduce 4-cycles, we analyze an equivalent problem: namely, when do four products of primes sum to 0 (each of the four summands may be taken with a positive or negative sign)? Combinatorial techniques allow us to derive conditions on sets of primes that either do or do not admit 4-cycles; the solution in some cases involve properties of twin and Sophie Germain primes. We additionally use a numerical approach to investigate the distribution of the sets of primes that admit 4-cycles and examine patterns that emerge.

Isomorphy class of trivolutions of $SL_2(K)$

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Trivolutions are group automorphisms of order 3. SL(2,K) is the group of 2-by-2 matrices with determinant 1. In this paper we will give a characterization for the isomorphy classes of trivolutions of SL(2,K) with K any field of characteristic not 2 or 3. This work is analogous to work by Helminck and Wu on automorphisms of order 2.

On Some Inverse Free Boundary Problems for Second Order Parabolic PDE's

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We consider the inverse free boundary problem, so called inverse Stefan problem (ISP), for a general second order linear parabolic PDE. The problem arises when considering phase transition processes with unknown temperature function and phase transition boundaries along with source term or boundary heat flux. We follow a new variational formulation developed recently in *U. G. Abdulla, Inverse Problems and Imaging, 7,2(2013),307-340*, and reformulate ISP as an optimal control problem for the minimization of the L_2 declination of traces of a state vector with the unknown flux and the free boundary as controls. This formulation has the benefits of reducing the effects of measurement errors, and the need to solve only a Neumann problem at each step of the minimization process. To prove existence of and provide a numerical method for solution of ISP, we consider a fully discrete problem. We prove well-posedness of the problem in the Sobolev spaces framework, and that interpolated solutions of the discrete problems converge weakly in Sobolev-Hilbert space H^1 to a solution to the Neumann problem via the derivation of two energy estimates for the discrete problem. We further prove that the discrete optimal control problems converge to the continuous problem with respect to functional and with respect to control. The numerical method is implemented, and numerical results are also presented.

On the Density of Ranges of Generalized Divisor Functions

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For a real number t, we may define a generalized divisor function σ_t by $\sigma_t(n) = \sum_{0<d|n} d^t$ for all positive integers n. It has been shown that the range of the function σ_{-1} is a dense subset of the interval $[1, \infty)$. For $r > 1$, it is not difficult to show that the range of σ_{-r} is a subset of the interval $[1, \zeta(r))$, where ζ denotes the Riemann zeta function. However, we may easily show that the range of σ_{-2} is not dense in the interval $[1, \zeta(2))$. We define a constant $\eta \approx 1.8877909$ and show that if $r > 1$, then the range of the function σ_{-r} is dense in the interval $[1, \zeta(r))$ if and only if $r \leq \eta$. If time permits, we will show how similar techniques may be used to solve related problems. For example, we may consider generalizations of unitary divisor functions or alternating divisor functions.

Hole Placement and Klein Links

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Klein links are links created by traversing strands along the surface of a once punctured Klein bottle. We introduce Klein links and their construction using a non-orientable rectangular diagram, a similar cousin to the class of torus knots and links. By examining how the hole on the diagram, where the Klein bottle punctures itself, affects the resulting Klein link, we modify the Klein link notation to incorporate the position of the hole. For the two possible cases of non-standard hole placement, two different theorems are created that each translate the hole closer to standard position. Then, a generalized formula that works for both cases is presented. Thus, with certain restrictions, we relate Klein links with non-standard hole placements to Klein links with the hole in standard position.

A Riemannian Approach to Optimization Over Compact Lie Groups

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We explore the problem of optimizing real-valued functions defined on compact Lie groups, a problem of increasing interest in mathematical optimization, using a plethora of tools from Riemannian geometry. After a brief prolegomenon to the necessary machinery and a laconic nod to finite groups, a new algorithm for maximization over connected compact Lie groups, based on classical gradient descent, is introduced alongside other optimization methods designed for real-valued functions on Riemannian manifolds. Then, we extend these results to the case of compact Lie groups that are not necessarily connected. Finally, the use of the developed methods to demonstrate the existence of Hadamard matrices of a given order, one of numerous possible applications, is briefly discussed.

Combinatorial proof of a q-series identity

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The use of generating functions and graphical representations of integer partitions can yield simple combinatorial proofs of some q-series identities. Particularly, the concept of Durfee square leads to sums involving quadratic polynomials as exponents. By employing these powerful methods, I obtained a combinatorial proof of the following formula $\sum_{n=0}^{\infty} \frac{q^{n^2-2n}a^n b^n}{\prod_{k=0}^{n-1}(1-aq^k)(1-bq^k)} = 1 + \frac{b}{q}\sum_{n=1}^{\infty} \frac{a^n}{\prod_{k=0}^{n-1}(1-bq^k)}$. After some further research, I found that this is a special case of the Rogers-Fine identity, one of the most important identities in the subject. It is remarkable that the left side of the formula is symmetric with respect to the parameters. This observation yields a corollary that has an interesting interpretation in terms of integer partitions.

Fredholm determinants and vanishing of L-functions at the central point

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Becauase the Riemann zeta function can be expressed as product over primes, we can pass from information about the zeros of the zeta function to information about the primes. The zeta function is the simplest example of an L-function. In general, one can construct L-fucntions associated to a wide range of objects. The zeros of these L-functions give important information about the object from which the L-function originated. In particular, finding the degree of vanishing of L functions at the point $s = 1/2$ is of principal importance in the study of the primes, modular forms, and ranks of the group of rational solutions of elliptic curves. We may garner information about the vanishing order by evaluating the sum of a rapidly decaying test function at the appropriately normalized imaginary part of these zeros. This is known as the 1-level density. There are analogous formulae for higher-level density. And, at the cost of computational complexity, higher-level densities provide better results on non-vanishing. The Katz-Sarnak density conjecture posits that for a fixed test function, the limiting average of the n-level density across a nice family of L-functions depends only on a symmetry group attached to the family. Confirmed for holomorphic cusp forms of weight k and level N (squarefree) for suitably restricted test functions, this result allows us to compute densities without explicitly knowing the zeros of L-functions.

Iwaniec, Luo, and Sarnak computed the 1-level density for families of holomorphic cusp functions. They found the optimal test functions to analyze the order of vanishing at the central point. Their results apply to test functions whose Fourier transforms are restricted to being supported in $(-2, 2)$. We generalize their analysis of Fredholm operators to find optimal test functions where the support is now any finite interval. Additionally, we determine the optimal test functions for the 2 and higher level densities, which allow us to obtain better results on the order of vanishing of L-functions at the central point.

Benfordness of Zeckendorf Decomposition

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We report on connections we have established between Zeckendorf decompositions and Benford's law. Zeckendorf showed that every positive integer can be decomposed uniquely into a sum of non-consecutive Fibonacci numbers; this result has been extended to decompositions arising from many other recurrence relations. Additionally, the Fibonacci numbers are known to satisfy Benford's law of digit bias, which means that the density of elements with first digit d is $\log_{10}(1+\frac{1}{d})$. According to this law, the smaller the digit, the more likely it is to occur as a leading digit. Thus the number 1 occurs as a leading digit about 30% of the time, while the number 9 occurs about 4.5% of the time. This phenomenon is of significant interest in pure mathematics, as well as in applications such as detecting tax fraud.

We prove that for a randomly selected integer between 1 and the nth Fibonacci number, as $n \to \infty$ the leading digits of the Fibonacci summands in its Zeckendorf decomposition are arbitrarily close to Benford almost surely. The proof proceeds by first analyzing random subsets of Fibonacci numbers for Benfordness. The main ingredient there is showing sets of density are preserved under this process. Using this, we solve our stated problem by proving a correspondence between Zeckendorf decompositions and random subsets of Fibonacci numbers. In those sets the Fibonacci numbers are chosen with a probability $p = 1/\varphi^2$ (where φ is the golden mean) if the previous Fibonacci number wasn't chosen, and 0 otherwise. We discuss generalizations to other recurrence relations as time permits.

Intrinsically Planar Linking

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A *planar n-link* is a disjoint collection of n 0-spheres and 1-spheres, embedded in the plane. We say a planar n-link, ℓ, is *split* if there exists a S^1 embedded in $\mathbb{R}^2 - \ell$ that bounds only non-trivial, complete, pieces of ℓ, on each side. Otherwise, we say ℓ is *non-split*. We say that a planar embedding of a graph, G is planar n-linked if it contains a non-split planar n-link.We begin with a characterization of *intrinsically planar 2-linked graphs*; that is planar graphs that have every planar embedding being planar linked. We have found the complete set of minor minimal intrinsically planar 2-linked graphs to be $\{K_4 \cup \{v\}, K_{3,2} \cup \{v\}, K_{3,1,1}\}$. In addition, we also exhibit several families of graphs that are minor minimal intrinsically planar n-linked for $n > 2$.

Finite conductor models for twists of GL(2) L-functions

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The Katz-Sarnak philosophy is that properties of families of L-functions are well modeled by corresponding quantities from ensembles of the classical compact groups; in particular, the spacing statistics of the zeros are similar in the limit to the spacings between eigenvalues of matrices. While these predictions have been verified and proved for many quantities in various families, S.J. Miller discovered a significant disagreement for finite conductors in the numerical data for elliptic curve L-functions. He observed a repulsion of their zeros away from the center of the critical strip that increased with the rank of the curve and decreased as the conductor increased. As the standard random matrix models poorly explained the observed behavior, he and his colleagues developed the Excised Orthogonal Model (EOM). Using the discretization of the L-functions at the central point and lower order terms in the 1-level density of these zeros (which incorporate the arithmetic structure of the curves), the zeros are instead compared to eigenvalues of large random orthogonal matrices whose characteristic polynomials exceed a certain cutoff value at 1 (the size of the matrices and the cutoff parameter depend on the arithmetic of the family of curves). Unlike the standard model, the EOM finds great agreement with both numerical data and pre-existing theory in the limit.

Since by the Modularity Theorem every elliptic curve L-function coincides with an L-function corresponding to a weight-two modular form, the EOM for elliptic curve L-functions restricts itself to modular L-functions attached to newforms of weight 2. We remove this restriction and consider the family of L-functions attached to quadratic twists of a fixed primitive holomorphic cusp form of arbitrary weight and level. This subsumes and extends the elliptic curve case. Moreover, since the discretization at the central point of a modular L-function is inversely proportional to the weight of the corresponding form, we prove that our model captures the disappearance of the repulsion at the central point for modular L-functions in the limit as the weight k tends to infinity. This limiting behavior was not observed in earlier studies, since all elliptic curves L-functions are modular of weight 2 and therefore have maximum discretization at the central point. The main ingredients in the proof are extending previous calculations to isolate the dependence on the arithmetic parameters in the family, especially the discretization of the L-functions at the central point. This requires a careful analysis of lower order terms in the statistical behavior of the zeros in order to determine the proper normalizations for finite conductors.

Degree Minimal Buchberger Graphs for Strongly Generic Monomial Ideals

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The free resolutions of monomial ideals in three variables over the ring $\Bbbk[x, y, z]$ can be described by a graph (called the *Buchberger graph* of a monomial ideal) and a simplicial or cellular complex. In the paper *Planar Graphs and Monomial Ideals* by Ezra Miller (Duke University) and Bernd Sturmfels (UC Berkeley), the authors show that every planar graph G (with a special property called 3-*connectedness*) can be realized as the 1-skeleton of a cellular resolution of some trivariate monomial ideal M_G. It remained an open question, however, how to minimally *construct* a monomial ideal with a resolution corresponding to a particular planar graph.

In our work this summer, we partially answer the question of how to construct such a monomial ideal. We enumerated the graphs G that can arise as the Buchberger graphs of monomial ideals which are *strongly generic* (given any pair of generating monomials m and m', if a variable divides both generators it must appear to different powers) and Artinian (the ideal contains x^n, y^m, z^k for some $n, m, k \in \mathbb{N}$,) and bounded the number of such graphs in terms of the degree n, m, k. We also provide an explicit algorithm which, given a triangulation of a 2-simplex with only three boundary vertices (and all other vertices on the interior), provides a monomial ideal with a free resolution supported on that simplicial complex.

While classifying possible Buchberger graphs of strongly generic monomial ideals, we provide a characterization of "dense" monomial ideals, where a monomial ideal M is *dense* if $x^{n+1}, y^{n+1}, z^{n+1}$ are in the generators of M for some n, and M has n "interior" minimal generators of the form $m_i = x^{\sigma_i} y^{\tau_i} z^i$. This characterization gives some insight into the size of principal ideals in the weak Bruhat order on permutations $\sigma \in S_n$.

Computational Results for Symmetric Chromatic Polynomials of Trees

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In a 1995 paper Richard Stanley defined χ_G, the symmetric chromatic polynomial of a Graph $G = (V, E)$. He then conjectured that χ_G distinguishes trees; a conjecture which still remains open. χ_G can be represented as a certain collection of integer partitions of $|V|$ induced by each $S \subseteq E$, which is very approachable with the aid of a computer. Our research involved writing a computer program for efficient verification of this conjecture for trees up to 23 vertices, as well as storing trees which have nearly identical symmetric chromatic polynomials. We then used this data to aid in formulating and proving necessary and sufficient conditions for trees to have matching subsections in their partition representations of their symmetric chromatic polynomials.

Arithmetic Properties of Infinite Products

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The work discussed here develops methods to evaluate certain infinite products in closed form. These are finite products of values of the Gamma function. Presented here are infinite products of rational functions R(n) raised to a power a(n). The sequences satisfy certain regularity conditions as either a k-periodic or k-automatic. Of particular interest is the

Regular Paperfolding sequence considered by J. P. Allouche. Also included are some results on the p-adic valuation of partial products of these types which also contain predictable patterns. Connections to the Alternating Sign Matrix sequence have appeared.

Gaussian Behavior of Generalized Zeckendorf Decompositions over Small Scales

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A beautiful theorem of Zeckendorf states that every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers $\{F_n\}$, with initial terms $F_1 = 1$, $F_2 = 2$. We consider the distribution of the number of summands involved in such decompositions. Previous work proved that as $n \to \infty$ the distribution of the number of summands in the Zeckendorf decompositions of $m \in [F_n, F_{n+1})$, appropriately normalized, converges to the standard normal. The proofs crucially used the fact that all integers in $[F_n, F_{n+1})$ share the same potential summands.
We generalize these results to subintervals of $[F_n, F_{n+1})$ as $n \to \infty$; the analysis is significantly more involved here as different integers have different sets of potential summands. Explicitly, fix an integer sequence $\alpha(n) \to \infty$. As $n \to \infty$, for almost all $m \in [F_n, F_{n+1})$ the distribution of the number of summands in the Zeckendorf decompositions of integers in the subintervals $I_{m;n} := [m, m + F_{\alpha(n)})$, appropriately normalized, converges to the standard normal. The proof follows by showing that, with probability tending to 1, m has at least one appropriately located large gap between indices in its decomposition. We then use a correspondence between this interval $I_{m;n}$ and $[0, F_{\alpha(n)})$ to obtain the result, since the summands are known to have Gaussian behavior in the latter interval. We also prove the same result for more general linear recurrences.

Construction of a New Family of Triangular Fractals

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The Sierpinski Gasket (SG) is generated by three contraction mappings with fixed points at the vertices of an equilateral triangle, and contraction ratio 1/2. We introduce a fourth contraction mapping with fixed point at the center of the equilateral triangle, and allow the contraction ratios to vary. We obtain a family of fractals that are far more complicated than the Sierpinski Gasket [1].

In the first part of this project, we investigate appropriate contraction ratios (α for the corner mappings and β for the central mapping) for the fractal to be connected and finitely-ramified. The appropriate pairs of α and β values are plotted on an α-β plane, and the points form a fractal curve in the plane. On one side of the curve, the values of α and β result in totally-disconnected fractals, while on the other side, the fractals are connected but not fintely-ramified.

The second part of this project focuses on the special case where $\alpha = 1/2$ and $\beta = 1/4$. Similar to Kigami's work on the Sierpinski Gasket, we first develop a harmonic extension algorithm on this special 4-gasket. Then we use the harmonic extension algorithm to construct the energy and Laplacian on the fractal.

[1] A finitely-ramified 4-gasket, `https://dl.dropboxusercontent.com/u/98351965/gasket.pdf`.